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Constant Riesz potentials on a circle in a plane with an application to polarization optimality problems

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ABSTRACT: A characterization for a Riesz s-potential function of a multiset ω_N of N points in $\mathbb{R}^2$ is given when $s = 2-2N$ and the potential function is constant on a circle in $\mathbb{R}^2$. The characterization allows us to partially prove a conjecture of Nikolov and Rafailov that if the potential function is constant on a circle Γ then the points in ω_N should be equally spaced on a circle concentric to Γ. As an application of constant Riesz s-potential functions, we also find all maximal and minimal polarization constants and configurations of two concentric circles in $\mathbb{R}^2$ for certain values of s.

KEYWORDS: roots of unity, max-min and min-max problems

MSC2010: 52A40

INTRODUCTION

For a fixed multiset of N points $\omega_N := \{x_1, x_2, \dots, x_N\}$ in $\mathbb{R}^2$ and a given constant $s \in \mathbb{R}$, we define the Riesz potential function $U^s(\cdot;\omega_N) : \mathbb{R}^2 \longrightarrow [0,\infty]$ as

$$U^s(x;\omega_N) := \sum_{j=1}^{N} |x - x_j|^{-s},$$

where $x \in \mathbb{R}^2$ and $|\cdot|$ is the 2-dimensional Euclidean norm in $\mathbb{R}^2$. We call $U^s(\cdot;\omega_N)$ a *Riesz s-potential function* of ω_N. See Ref. 1 for more information on Riesz s-potential functions in a d-dimensional Euclidean space $\mathbb{R}^d$.

In this paper, we consider two problems concerning the Riesz s-potential functions $U^s(\cdot;\omega_N)$. The first problem is to prove, in parts, Nikolov and Rafailov's conjecture about points in ω_N being equally spaced on some circle when a Riesz s-potential function is constant. The second problem is to solve polarization optimality problems when this Riesz s-potential function is constant.

Let ω_N be a fixed set of distinct equally spaced points on a circle $T \subseteq \mathbb{R}^2$ and Γ be a circle concentric to T. In Ref. 2, Nikolov and Rafailov show in Theorem 1 that $U^s(x;\omega_N)$ is constant as a function of x on Γ if and only if $s \in \{0, -2, -4, \dots, 4-2N, 2-2N\}$. They also show in Theorem 2 that this gives a characteristic property of distinct equally spaced points on a circle. More precisely, given a set ω_N of N distinct points such that $U^s(x;\omega_N)$ is constant on a circle Γ for every $s \in \{-2, -4, \dots, 2-2N\}$ (the constant may depend on s), the points in ω_N are equally spaced on some circle concentric to Γ. In the same paper, it was conjectured (Conjecture 2) that only $s = 2-2N$ should be sufficient. We state the conjecture below.

Conjecture 1 *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{R}^2$ and a circle $\Gamma \subseteq \mathbb{R}^2$ such that*

$$U^{2-2N}(x;\omega_N) = \sum_{j=1}^{N} |x - x_j|^{2N-2}$$

is constant as a function of x on Γ. Then ω_N forms a set of distinct equally spaced points on a circle concentric to Γ.

The conjecture was verified in the case $N = 3$ (see Ref. 2, Proposition 2). In this paper, we prove Conjecture 1 in the following cases (after translating the centre of Γ to the origin):

(i) when all points $x_1, x_2, \dots, x_N$ have the same norm (Proposition 1);

(ii) when $N=4$ and x_1, x_2, x_3, x_4 have an equal angle distribution (Proposition 2);

(iii) when N is prime and $x_1, x_2, \dots, x_N$ have an equal angle distribution and rational norms (Proposition 3).

The above results are based on a characterization of ω_N when $U^{2-2N}(\cdot;\omega_N)$ is constant on the unit circle (Theorem 1).

The next problems considered in this paper are polarization optimality problems. Let $\omega_N = \{x_1, \dots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^2$. Denote by

$$\mathbb{S}^1_R := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centred at the origin of radius R. When $R=1$, we simply use the notation $\mathbb{S}^1$. Given $s \in \mathbb{R}$, $R>0$, and $r>0$, we define polarization constants

$$M^s_N(\mathbb{S}^1_r;\mathbb{S}^1_R) := \max_{\substack{\omega_N \subseteq \mathbb{S}^1_r \\ \#\omega_N = N}} \min_{y \in \mathbb{S}^1_R} U^s(y;\omega_N), \tag{1}$$

$$M^0_N(\mathbb{S}^1_r;\mathbb{S}^1_R) := N,$$

$$m^s_N(\mathbb{S}^1_r;\mathbb{S}^1_R) := \min_{\substack{\omega_N \subseteq \mathbb{S}^1_r \\ \#\omega_N = N}} \max_{y \in \mathbb{S}^1_R} U^s(y;\omega_N), \tag{2}$$

$$m^0_N(\mathbb{S}^1_r;\mathbb{S}^1_R) := N,$$

where $\#\omega_N$ denotes the cardinality of the multiset ω_N. We will call ω_N a *maximal (minimal) N-point Riesz s-polarization configuration* of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$ if ω_N attains the maximum in (1) (minimum in (2)). We give a brief history of such polarization optimality problems below.

Farkas and Révész[3] were the first to introduce two-plate polarization constants in a general sense. However, all previous results[4–6] on polarization optimality problems related to Riesz potentials were considered for the case when $R=r=1$. The maximality of N distinct equally spaced points on the unit circle for the maximal Riesz s-polarization problem of $(\mathbb{S}^1;\mathbb{S}^1)$ in (1) was proved in Ref. 4 for $s=2$. Erdélyi and Saff[1] established this for $s=4$. For arbitrary $s>0$, this result was proved in Ref. 5 where they also showed the minimality of N distinct equally spaced points on the unit circle for the minimal Riesz s-polarization problem of $(\mathbb{S}^1;\mathbb{S}^1)$ in (2) for $-1 \leqslant s < 0$. Note that minimal N-point Riesz s-polarization problems of $(\mathbb{S}^1;\mathbb{S}^1)$ when $s>0$ are not interesting because $m^s_N(\mathbb{S}^1;\mathbb{S}^1) = \infty$ for all $s>0$.

Up to the present, there are no results on polarization optimality problems in (1) and (2) for $R \neq r$. In this paper, we give a characterization of all maximal and minimal N-point Riesz s-polarization configurations of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$ when $s = -2, -4, \dots, 2-2N$.

Although the asymptotic properties of polarization constants are not our main interest in this paper, it is worth mentioning the asymptotic types of behaviour of $M^s_N(\mathbb{S}^1;\mathbb{S}^1)$ as $N \to \infty$ [5]:

$$M^s_N(\mathbb{S}^1;\mathbb{S}^1) \sim \begin{cases} \dfrac{2\zeta(s)}{(2\pi)^s}(2^s-1)N^s, & s>1, \\ (1/\pi)N\log N, & s=1, \\ \dfrac{2^{-s}}{\sqrt{\pi}}\dfrac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}N, & 0 \leqslant s < 1, \end{cases}$$

where $\zeta(s)$ denotes the classical Riemann zeta function and $a_N \sim b_N$ means that $\lim_{N\to\infty} a_N/b_N = 1$. The reader is referred to Refs. 1, 7, 8 for asymptotic results of polarization constants and configurations of general subsets of $\mathbb{R}^d$ as $N \to \infty$ when $s>0$.

CONSTANT RIESZ s-POTENTIAL FUNCTIONS

The Euclidean space $\mathbb{R}^2$ and the complex space $\mathbb{C}$ over $\mathbb{R}$ have the same dimension and the same norm. However, the complex space $\mathbb{C}$ has a richer algebraic structure; for example, $\mathbb{C}$ is a field. Hence when we prove all theorems in this and the next section, any element $x \in \mathbb{R}^2$ will be replaced by $x \in \mathbb{C}$, the 2-dimensional Euclidean norm $|\cdot|$ is replaced by the modulus in $\mathbb{C}$, and the notation xy is adopted from the multiplication in $\mathbb{C}$ and the notation x/y is adopted from the division in $\mathbb{C}$. We recall that the usual dot product in $\mathbb{C}$ is defined by

$$(a_1 + a_2\mathrm{i}) \cdot (b_1 + b_2\mathrm{i}) := a_1 b_1 + a_2 b_2.$$

Now let $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct points. In this section, we will assume that $U^{2-2N}(x;\omega_N) = \sum_{j=1}^N |x-x_j|^{2N-2}$ is constant (as a function of x) on a circle $\Gamma \subset \mathbb{C}$ and prove that, under various conditions, the points $x_1, x_2, \dots, x_N$ are equally spaced on some circle concentric to Γ. By translating and scaling the circle Γ, we can assume without loss of generality that Γ is the unit circle $\mathbb{S}^1$. The following conjecture is equivalent to Conjecture 1.

Conjecture 2 *Given a set of N distinct points $\omega_N := \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ such that*

$$U^{2-2N}(x;\omega_N) = \sum_{j=1}^N |x-x_j|^{2N-2}$$

is constant as a function of x on $\mathbb{S}^1$, then ω_N forms a set of distinct equally spaced points on $\mathbb{S}^1_R$ for some R.

We begin with our main theorem which gives a characterization of ω_N when $U^{2-2N}(\cdot;\omega_N)$ is constant on the unit circle.

Theorem 1 *Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct points. Then the function*

$$U^{2-2N}(x;\omega_N) = \sum_{j=1}^{N} |x - x_j|^{2N-2}$$

is constant on the circle $\mathbb{S}^1$ if and only if

$$\sum_{j=1}^{N}\sum_{q=0}^{N-k-1}\binom{N-1}{q}\binom{N-1}{k+q}|x_j|^{2N-2k-2q-2}x_j^k = 0, \qquad \text{for all} \quad k = 1,\dots,N-1. \tag{3}$$

Note that (3) gives a system of $N-1$ equations in terms of elements in the set ω_N. The proof of Theorem 1 requires a technical lemma which involves a lot of calculations, and so we will postpone it to the end of this section.

Example 1 Suppose $U^{2-2N}(x;\omega_N)$ is constant on $\mathbb{S}^1$. We list the systems of equations (3) that the x_j must satisfy for small values of N below.
(i) Let $N = 3$. Then x_1, x_2, x_3 must satisfy

$$\sum_{j=1}^{3} x_j^2 = 0, \qquad \sum_{j=1}^{3}(1+|x_j|^2)x_j = 0.$$

(ii) Let $N = 4$. Then x_1, x_2, x_3, x_4 must satisfy

$$\sum_{j=1}^{4} x_j^3 = 0, \qquad \sum_{j=1}^{4}(1+|x_j|^2)x_j^2 = 0,$$
$$\sum_{j=1}^{4}(1+3|x_j|^2+|x_j|^4)x_j = 0.$$

(iii) Let $N = 5$. Then x_1, x_2, x_3, x_4, x_5 must satisfy

$$\sum_{j=1}^{5} x_j^4 = 0, \qquad \sum_{j=1}^{5}(1+|x_j|^2)x_j^3 = 0,$$
$$\sum_{j=1}^{5}(3+8|x_j|^2+3|x_j|^4)x_j^2 = 0,$$
$$\sum_{j=1}^{5}(1+5|x_j|^2+|x_j|^4)(1+|x_j|^2)x_j = 0.$$

Using the characterization given in Theorem 1, we can verify Conjecture 2 in various cases. Our first result asserts that Conjecture 2 holds if the points $x_1, x_2, \dots, x_N$ already lie on the same circle centred at the origin (i.e., they have the same norm).

Proposition 1 *Let $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{C}$ be a set of N distinct non-zero points lying on some circle centred at the origin. If $U^{2-2N}(\cdot;\omega_N)$ is constant on $\mathbb{S}^1$, then $x_1, x_2, \dots, x_N$ are equally spaced.*

Proof: It suffices to show that $x_1, x_2, \dots, x_N$ are the Nth roots of some complex number. Suppose $|x_1| = |x_2| = \cdots = |x_N| = R$. From (3) we deduce that

$$\sum_{j=1}^{N} x_j^k = 0,$$

for all $k = 1, 2, \dots, N-1$. By Newton's identities,

$$e_k(x_1, x_2, \dots, x_N) = 0, \qquad k = 1, 2, \dots, N-1,$$

where the e_k are elementary symmetric polynomials. Thus $x_1, x_2, \dots, x_N$ are distinct roots of the polynomial

$$\prod_{k=1}^{N}(X - x_k) = X^N - \mu$$

for some $\mu \in \mathbb{C}$. □

Now we will consider another special case. Instead of assuming that all points have the same norm, we will assume that they have an equal angle distribution around the origin. More precisely, let $\zeta = \mathrm{e}^{2\pi\mathrm{i}/N}$ and, without loss of generality, we assume that

$$x_1 = r_1\zeta^1,\ x_2 = r_2\zeta^2,\ \dots,\ x_N = r_N\zeta^N \tag{4}$$

for some positive real numbers $r_1, r_2, \dots, r_N$. Our next result proves Conjecture 2 when $N = 4$ and x_1, x_2, x_3, x_4 have an equal angle distribution.

Proposition 2 *Let x_1, x_2, x_3, x_4 be as in (4). Suppose that*

$$U^{-6}(x;\omega_N) := \sum_{j=1}^{N} |x - x_j|^6$$

is constant as a function of x on $\mathbb{S}^1$. Then x_1, x_2, x_3, x_4 are equally spaced on a circle centred at the origin.

Proof: By Proposition 1, it suffices to show that $|x_1| = |x_2| = |x_3| = |x_4|$. From Example 1, the points

x_1, x_2, x_3, x_4 must satisfy

$$\sum_{j=1}^{4} x_j^3 = \sum_{j=1}^{4}(1+|x_j|^2)x_j^2$$
$$= \sum_{j=1}^{4}(1+3|x_j|^2+|x_j|^4)x_j = 0.$$

With $x_j = r_j\zeta^j$, the equation $\sum_{j=1}^4 x_j^3 = 0$ becomes

$$r_1^3\zeta^3 + r_2^3\zeta^2 + r_3^3\zeta + r_4^3 = 0.$$

Let $P(X) = r_1^3X^3 + r_2^3X^2 + r_3^3X + r_4^3 \in \mathbb{R}[X]$. Since $\zeta = \mathrm{i}$, $\bar{\zeta} = -\mathrm{i}$ are roots of $P(X)$, we have

$$P(X) = C(X^2+1)(X+b) = C(X^3+bX^2+X+b),$$

for some non-zero $C \in \mathbb{R}$. Comparing the coefficients, we have $r_1 = r_3$, $r_2 = r_4$.

The equation $\sum_{j=1}^4(1+|x_j|^2)x_j^2 = 0$ becomes $\sum_{j=1}^4(1+r_j^2)r_j^2\zeta^{2j} = 0$. Expanding the sum and using $r_1 = r_3$, $r_2 = r_4$ we have

$$2(1+r_1^2)r_1^2\zeta^2 + 2(1+r_2^2)r_2^2 = 0.$$

Since $\zeta^2 = -1$ we obtain $(1+r_1^2)r_1^2 = (1+r_2^2)r_2^2$. Let $t = r_2/r_1$ and $a = 1/r_1^2$. We have

$$(a+1) = (a+t^2)t^2 \implies t^4 + at^2 - (a+1) = 0.$$

Thus

$$t^2 = \frac{-a \pm \sqrt{a^2+4a+4}}{2} = \frac{-a \pm (a+2)}{2}.$$

The only possible case is $t^2 = \frac{1}{2}(-a+(a+2)) = 1$. Since $t > 0$ we have $t = 1$. Hence $r_2 = r_1$. We have shown that $r_1 = r_2 = r_3 = r_4$. □

Actually, if we further assume that all norms are rational, then Conjecture 2 holds for all prime N.

Proposition 3 *Let N be a prime number. Let $x_1, x_2, \dots, x_N$ be as in (4) where all $r_j \in \mathbb{Q}$. Suppose that $U^{2-2N}(\cdot;\omega_N)$ is constant on $\mathbb{S}^1$. Then $x_1, x_2, \dots, x_N$ are equally spaced on a circle centred at the origin.*

Proof: By Proposition 1, it suffices to show that $|x_1| = |x_2| = \cdots = |x_N|$. Applying the condition (3) with $k = N-1$ gives $\sum_{j=1}^N x_j^{N-1} = 0$. Thus

$$\sum_{j=1}^{N} r_j^{N-1}\zeta^{-j} = \sum_{j=1}^{N} r_{N-j}^{N-1}\zeta^j = 0.$$

Let A be a positive integer so that $Ar_{N-j}^{N-1} \in \mathbb{Z}_{>0}$ for every j. Then $\sum_{j=1}^N (Ar_{N-j}^{N-1})\zeta^k = 0$. This is a vanishing linear combination of $1, \zeta, \dots, \zeta^{N-1}$ with positive-integer coefficients. Since the minimal polynomial of ζ is $1 + X + \cdots + X^{N-1}$ (N is prime), this implies that all coefficients are equal. Thus $Ar_1^{N-1} = Ar_2^{N-1} = \cdots = Ar_N^{N-1}$ and hence $r_1 = r_2 = \cdots = r_N$. □

Proof of Theorem 1

The following technical lemma is needed for the proofs of Theorem 1 and Theorem 2.

Lemma 1 *Let $N \in \mathbb{N}$ and $p \in \{1, 2, \dots, N-1\}$ be fixed. If $x_j := |x_j|\cos t_j + \mathrm{i}|x_j|\sin t_j$ for all $j = 1, 2, \dots, N$, then for all $y := \cos t + \mathrm{i}\sin t \in \mathbb{S}^1$,*

$$\sum_{j=1}^{N}|y - x_j|^{2p}$$
$$= E_0 + \sum_{k=1}^{p}\sum_{j=1}^{N}[E_{k,j}\cos kt_j \cos kt + E_{k,j}\sin kt_j \sin kt], \tag{5}$$

where

$$E_0 = \sum_{j=1}^{N}\sum_{q=0}^{p}\binom{p}{q}^2 |x_j|^{2p-2q},$$
$$E_{k,j} = (-1)^k 2\sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q}|x_j|^{2p-k-2q}.$$

Proof: Let $y := \cos t + \mathrm{i}\sin t \in \mathbb{S}^1$ and $x_j := |x_j|\cos t_j + \mathrm{i}|x_j|\sin t_j$ for all $j = 1, 2, \dots, N$. A simple calculation shows that

$$f_j(t) := |y - x_j|^{2p} = (|x_j|^2 + 1 - 2|x_j|\cos(t - t_j))^p.$$

Since $A := \{1, \cos(t-t_j), \dots, \cos p(t-t_j)\}$ forms an orthogonal system with respect to the inner product

$$\langle f, g\rangle := \int_0^{2\pi} f(t)g(t)\,\mathrm{d}t$$

and $f_j \in \operatorname{span}(A)$, we have

$$f_j(t) = \sum_{k=0}^{p} E_{k,j}\cos k(t - t_j) = E_{0,j}$$
$$+ \sum_{k=1}^{p} E_{k,j}(\cos kt_j \cos kt + \sin kt_j \sin kt).$$

Now,

$$\sum_{j=1}^{N}|y-x_j|^{2p} = \sum_{j=1}^{N} f_j(t)$$
$$= E_0 + \sum_{k=1}^{p}\sum_{j=1}^{N}[E_{k,j}\cos kt_j \cos kt + E_{k,j}\sin kt_j \sin kt],$$

where $E_0 = \sum_{j=1}^{N} E_{0,j}$. By the orthogonality of the elements in the set A and the calculation in Lemma 3 in the last section, we have

$$E_0 = \sum_{j=1}^{N}\frac{\langle f_j, 1\rangle}{2\pi} = \sum_{j=1}^{N}\sum_{q=0}^{p}\binom{p}{q}^2 |x_j|^{2p-2q}$$

and

$$E_{k,j} = \frac{\langle f_j, \cos k(t-t_j)\rangle}{\pi}$$
$$= (-1)^k 2\sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q}|x_j|^{2p-k-2q},$$

for all $k\in\{1,2,\dots,p\}$ and $j\in\{1,2,\dots,N\}$. □

Proof of Theorem 1: For each $j=1,2,\dots,N$, set

$$x_j := |x_j|\cos t_j + \mathrm{i}|x_j|\sin t_j.$$

(⇒) By our assumption, $f(y) := \sum_{j=1}^{N}|y-x_j|^{2N-2}$ is constant on $\mathbb{S}^1$, say $f(y)=C$ on $\mathbb{S}^1$. Set

$$y = \cos t + \mathrm{i}\sin t \in \mathbb{S}^1.$$

By (5) for all $t\in[0,2\pi]$,

$$C = f(y) = E_0 + \sum_{k=1}^{N-1}\sum_{j=1}^{N}[E_{k,j}\cos kt_j\cos kt + E_{k,j}\sin kt_j \sin kt]. \tag{6}$$

Because the set $\{1, \cos t, \sin t, \dots, \cos(N-1)t, \sin(N-1)t\}$ is linearly independent over $\mathbb{R}$, we deduce

$$C - E_0 = 0$$

and for all $k=1,2,\dots,N-1$,

$$\sum_{j=1}^{N}E_{k,j}\cos kt_j = 0 \quad\text{and}\quad \sum_{j=1}^{N}E_{k,j}\sin kt_j = 0. \tag{7}$$

Using the formulae of $E_{k,j}$ from Lemma 1, it follows from (7) that for all $k=1,2,\dots,N-1$,

$$0 = \sum_{j=1}^{N}E_{k,j}(\cos kt_j + \mathrm{i}\sin kt_j) = \sum_{j=1}^{N}\frac{E_{k,j}}{|x_j|^k}x_j^k$$
$$= (-1)^k 2\sum_{j=1}^{N}\sum_{q=0}^{N-k-1}\binom{N-1}{q}\binom{N-1}{k+q} \times |x_j|^{2N-2k-2q-2}x_j^k, \tag{8}$$

which implies (3).

(⇐) Assume that (3) holds. Then we have (8) and (7). Combining (7) and the second identity in (6), we have for all $y\in\mathbb{S}^1$,

$$\sum_{j=1}^{N}|y-x_j|^{2N-2} = E_0,$$

which implies that $U^{2-2N,h}(\cdot;\omega_N)$ is constant on $\mathbb{S}^1$. □

AN APPLICATION TO POLARIZATION OPTIMALITY PROBLEMS

We remind the reader that we will consider polarization optimality problems in the complex plane. A complete characterization of all maximal and minimal N-point Riesz s-polarization configurations of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$ when $s=-2,-4,\dots,2-2N$ is given as follows.

Theorem 2 *Let $N\in\mathbb{N}$, $p\in\{1,2,\dots,N-1\}$, $R>0$, $r>0$, and $\{x_1,x_2,\dots,x_N\}\subseteq\mathbb{S}^1_r$. The following statements are equivalent:*

(a) $\{x_1,x_2,\dots,x_N\}$ is a maximal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$;

(b) $\{x_1,x_2,\dots,x_N\}$ is a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}^1_r;\mathbb{S}^1_R)$;

(c) $\sum_{j=1}^{N}x_j = \sum_{j=1}^{N}x_j^2 = \dots = \sum_{j=1}^{N}x_j^p = 0$.

Furthermore,

$$M_N^{-2p}(\mathbb{S}^1_r;\mathbb{S}^1_R) = m_N^{-2p}(\mathbb{S}^1_r;\mathbb{S}^1_R) = \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2(2rR)^{2j}(r^2+R^2+|r^2-R^2|)^{p-2j}. \tag{9}$$

Unlike the case when $R=r=1$ and $s>0$, optimal configurations for the cases in Theorem 2 may not be unique up to rotation. For example, when $p=1$ and $N=4$, our characterization of optimal configurations is $\sum_{j=1}^{4}x_j=0$. Hence there are infinitely many optimal configurations that are not rotations of one another. The proof of Theorem 2 relies on the fact that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}^1_r$, then for each $s = -2, -4, \dots, 2-2N$, $U^s(\cdot, \omega_N)$ is constant on $\mathbb{S}^1_R$. This special property allows the problems to have more than one solution (up to rotation). Furthermore, our experimental study suggests that for the cases when $s \in \mathbb{R}^2 \backslash \{0, -2, -4, \dots, 2-2N\}$, any maximal and minimal N-point Riesz s-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$ is unique up to rotation, namely, it is a configuration of distinct equally spaced points on $\mathbb{S}^1_r$. We make the following conjecture.

Conjecture 3 *Let $N \in \mathbb{N}$, $s \in \mathbb{R} \backslash \{0, -2, -4, \dots, 2-2N\}$, $R > 0$, $r > 0$, and $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}^1_r$. The following statements are equivalent:*

(a) $\{x_1, x_2, \dots, x_N\}$ is a maximal N-point Riesz s-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$;

(b) $\{x_1, x_2, \dots, x_N\}$ is a minimal N-point Riesz s-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$;

(c) $\{x_1, x_2, \dots, x_N\}$ is a configuration of distinct equally spaced points on $\mathbb{S}^1_r$.

Proof of Theorem 2

We need the following lemma.

Lemma 2 *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, and $r > 0$. Then any configuration of N distinct equally spaced points on $\mathbb{S}^1_r$ is both a maximal and a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$.*

Proof: Let $\omega_N := \{x_1, x_2, \dots, x_N\}$ be a configuration of N distinct equally spaced points on $\mathbb{S}^1_r$ and $p \in \{1, 2, \dots, N-1\}$ be fixed. By Theorem 1 in Ref. 2, we know that $f(x) := \sum_{j=1}^N |x - x_j|^{2p}$ is constant as a function of x on $\mathbb{S}^1_R$, say $f(x) = C$ for all $x \in \mathbb{S}^1_R$. Thus

$$\max_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p} = C = \min_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p}. \quad (10)$$

Let $\{y_1, y_2, \dots, y_N\}$ be any N-point configuration on $\mathbb{S}^1_r$. To show that ω_N is a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$, we will show that

$$\max_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p} \leqslant \max_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |y_i - x|^{2p}. \quad (11)$$

Consider

$$\left| x_j - \frac{R}{y_i/r} \right| = \left| \frac{x_j}{y_i} \left(y_i - \frac{R}{x_j/r} \right) \right| = \left| y_i - \frac{R}{x_j/r} \right|.$$

As $R/(y_i/r) \in \mathbb{S}^1_R$ for all i, we have

$$NC = \sum_{i=1}^N f\left(\frac{R}{y_i/r} \right) = \sum_{i=1}^N \sum_{j=1}^N \left| x_j - \frac{R}{y_i/r} \right|^{2p} = \sum_{j=1}^N \sum_{i=1}^N \left| y_i - \frac{R}{x_j/r} \right|^{2p}. \quad (12)$$

It follows from (12) that there is $j_0 \in \{1, 2, \dots, N\}$ such that

$$C \leqslant \sum_{i=1}^N \left| y_i - \frac{R}{x_{j_0}/r} \right|^{2p} \leqslant \max_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |y_i - x|^{2p}.$$

But $C = \max_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p}$ from (10). Hence we have (11) as required.

To show that ω_N is a maximal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}^1_r; \mathbb{S}^1_R)$, we will show that

$$\min_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |y_i - x|^{2p} \leqslant \min_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p}. \quad (13)$$

It follows from (12) that there is $j_0' \in \{1, 2, \dots, N\}$ such that

$$\min_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |y_i - x|^{2p} \leqslant \sum_{i=1}^N \left| y_i - \frac{R}{x_{j_0'}/r} \right|^{2p} \leqslant C.$$

But $C = \min_{x \in \mathbb{S}^1_R} \sum_{i=1}^N |x_i - x|^{2p}$ from (10). Hence we have (13) as required. □

Proof of Theorem 2: Because the proof of (a) ⟺ (c) is similar to the proof of (b) ⟺ (c), we will show only (b) ⟺ (c) and skip the proof of (a) ⟺ (c). Without loss of generality, we can assume that $R = 1$.

Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, and $r > 0$ be fixed and $\{x_1, x_2, \dots, x_N\}$ be any configuration in $\mathbb{S}^1_r$. We recall from Lemma 1 that for $x_j := r\cos t_j + \mathrm{i} r \sin t_j$ for all $j = 1, 2, \dots, N$ and for all $y := \cos t + \mathrm{i} \sin t \in \mathbb{S}^1$,

$$\sum_{j=1}^N |y - x_j|^{2p} = E_0 + \sum_{k=1}^p \sum_{j=1}^N [E_{k,j} \cos kt_j \cos kt + E_{k,j} \sin kt_j \sin kt], \quad (14)$$

$$= E_0 + \sum_{k=1}^p \sum_{j=1}^N \left[\frac{E_{k,j}}{r^k} (y^k \cdot x_j^k) \right], \quad (15)$$

where

$$E_0 = \sum_{j=1}^{N}\sum_{q=0}^{p}\binom{p}{q}^2 r^{2p-2q},$$

$$\frac{E_{k,j}}{r^k} = (-1)^k 2\sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q} r^{2p-2k-2q}. \quad (16)$$

Notice that the constant E_0 does not depend on a configuration on $\mathbb{S}_r^1$ and the constants $E_{k,j}/r^k$ do not depend on a configuration on $\mathbb{S}_r^1$ and j. For convenience for all configurations $\{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$, we set

$$E_k := \frac{E_{k,j}}{r^k}, \qquad \text{for all } k = 1, 2, \dots, p. \quad (17)$$

First of all, we will show that

$$m_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}^1) = E_0.$$

Let $\omega_N' := \{x_1', x_2', \dots, x_N'\}$ be a configuration of distinct equally spaced points on $\mathbb{S}_r^1$. Using (15), we have for all $y \in \mathbb{S}^1$,

$$\sum_{j=1}^{N}|y - x_j'|^{2p} = E_0 + \sum_{k=1}^{p}\sum_{j=1}^{N} E_k (y^k \cdot (x_j')^k)$$

$$= E_0 + \sum_{k=1}^{p} E_k\left(y^k \cdot \sum_{j=1}^{N}(x_j')^k\right) = E_0 \quad (18)$$

where the last equality follows from the fact that $\sum_{j=1}^{N}(x_j')^k = 0$ for all $k = 1, 2, \dots, p$. Since ω_N' is a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$ (by Lemma 2), we obtain

$$m_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}^1) = \max_{y\in\mathbb{S}^1} U^{-2p}(y; \omega_N') = E_0.$$

We now prove (c)⇒(b). Assume that $\omega_N = \{x_1, x_2, \dots, x_N\} \subseteq \mathbb{S}_r^1$ such that $\sum_{j=1}^{N} x_j^k = 0$ for all $k = 1, 2, \dots, p$. Applying the same argument as in (18), we have for all $y \in \mathbb{S}^1$,

$$U^{-2p}(y; \omega_N) = E_0 + \sum_{k=1}^{p} E_k\left(y^k \cdot \sum_{j=1}^{N} x_j^k\right) = E_0,$$

which implies that ω_N is a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$.

Next, we show (b)⇒(c). Assume that $\omega_N = \{x_1, x_2, \dots, x_N\}$ is a minimal N-point Riesz $-2p$-polarization configuration of $(\mathbb{S}_r^1; \mathbb{S}^1)$. Then for all $y \in \mathbb{S}^1$,

$$U^{-2p}(y; \omega_N) = \sum_{j=1}^{N}|y - x_j|^{2p} \leqslant m_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}^1) = E_0.$$

Then, by (14) and (17) for all $t \in [0, 2\pi]$,

$$E_0 \geqslant U^{-2p}(y; \omega_N) = E_0 + \sum_{k=1}^{p}(\mathscr{C} \cos kt + \mathscr{S} \sin kt).$$

where $\mathscr{C} = \sum_{j=1}^{N} E_k \cos kt_j$ and $\mathscr{S} = \sum_{j=1}^{N} E_k \sin kt_j$. Thus for all $t \in [0, 2\pi]$,

$$0 \geqslant \sum_{k=1}^{p}(\mathscr{C} \cos kt + \mathscr{S} \sin kt).$$

Hence for all $t \in [0, 2\pi]$,

$$\sum_{k=1}^{p}(\mathscr{C} \cos kt + \mathscr{S} \sin kt) = 0.$$

Because $\{\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos pt, \sin pt\}$ is a linearly independent set over $\mathbb{R}$ for all $k = 1, 2, \dots, p$,

$$\sum_{j=1}^{N} E_k \cos kt_j = \sum_{j=1}^{N} E_k \sin kt_j = 0.$$

Since for all $k = 1, 2, \dots, p$, $E_k \neq 0$ ((16)),

$$\sum_{j=1}^{N} \cos kt_j = \sum_{j=1}^{N} \sin kt_j = 0, \qquad k = 1, 2, \dots, p,$$

which implies that $\sum_{j=1}^{N} x_j^k = \sum_{j=1}^{N} r^k(\cos kt_j + \mathrm{i} \sin kt_j) = 0$ for all $k = 1, 2, \dots, p$.

To compute $M_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}_R^1)$ and $m_N^{-2p}(\mathbb{S}_r^1; \mathbb{S}_R^1)$ in (9), we can use a similar argument in Lemma 1 by replacing $y = R\cos t + \mathrm{i}R\sin t$ and $f_j(t) = |y - x_j|^{2p} = (r^2 + R^2 - 2Rr\cos(t - t_j))^p$. Applying the calculations as in Lemma 4, it is not difficult to check that if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}_r^1$, then for all $y \in \mathbb{S}_R^1$,

$$U^{-2p}(y; \omega_N)$$
$$= \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2 (2rR)^{2j}(r^2 + R^2 + |r^2 - R^2|)^{p-2j}.$$

□

COMPUTATIONS OF INTEGRALS

We collect our computations of all integrals in this section.

Lemma 3 *Let $p \in \mathbb{N}$, $k \in \{0, 1, \dots, p\}$, and $z \in \mathbb{C}$. Then*

$$\int_0^{2\pi} (z^2 + 1 - 2z\cos t)^p \cos kt \,\mathrm{d}t$$
$$= (-1)^k 2\pi \sum_{q=0}^{p-k}\binom{p}{q}\binom{p}{k+q} z^{2p-k-2q}. \quad (19)$$

Proof: Let $p \in \mathbb{N}$ and $k \in \{0, 1, \dots, p\}$. First, we prove the equality (19) for $z \in \mathbb{R}$. Let $z \in \mathbb{R}$. Then, for $\zeta = \mathrm{e}^{\mathrm{i}t}$,

$$\begin{aligned}\int_0^{2\pi} &(z^2 + 1 - 2z\cos t)^p \cos kt \,\mathrm{d}t \\ &= \int_0^{2\pi} (z^2 + 1 - z(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}))^p \,\mathrm{e}^{\mathrm{i}kt} \,\mathrm{d}t \\ &= \int_0^{2\pi} (z - \mathrm{e}^{\mathrm{i}t})^p (z - \mathrm{e}^{-\mathrm{i}t})^p \,\mathrm{e}^{\mathrm{i}kt} \,\mathrm{d}t \\ &= \frac{1}{\mathrm{i}} \int_{\mathbb{S}^1} (z - \zeta)^p (z - 1/\zeta)^p \zeta^{k-1} \,\mathrm{d}\zeta \\ &= 2\pi \operatorname{Res}\left(\frac{(z-\zeta)^p (z\zeta - 1)^p}{\zeta^{p-k+1}}; \zeta = 0 \right) \\ &= (-1)^k 2\pi \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} z^{2p-k-2q},\end{aligned}$$

where the first equality follows from the fact that the last expression is a real number. Notice that the left-hand side and the right-hand side of (19) are polynomials as functions of z. Thus both functions are analytic on $\mathbb{C}$ and we have (19) for all $z \in \mathbb{C}$. □

Lemma 4 *Let $p \in \mathbb{N}$ and $k \in \{0, 1, \dots, p\}$. For $a, b \in \mathbb{C}$,*

$$\int_0^{2\pi} (a - b\cos t)^p \cos kt \,\mathrm{d}t = \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} C_{a,b,p,q,k}, \quad (20)$$

where $C_{a,b,p,q,k} = b^{2q+k}(a \pm \sqrt{a^2 - b^2})^{p-k-2q}$ and the square root function in (20) can be selected to be both branches of the complex square root function.

Proof: Clearly, if $b = 0$, then the equation in (20) is $0 = 0$. Assume that $b \in \mathbb{C}\backslash\{0\}$ and $a \in \mathbb{C}$. To reduce (20) to (19), we consider

$$(\lambda a - \lambda b \cos t)^p,$$

where λ is chosen to satisfy the equations

$$2z = b\lambda, \qquad z^2 + 1 = a\lambda,$$

for some $z \in \mathbb{C}$. From the above equations,

$$z = \frac{a \pm \sqrt{a^2 - b^2}}{b}$$

and

$$\lambda = \frac{2z}{b} = \frac{2a \pm 2\sqrt{a^2 - b^2}}{b^2}.$$

Furthermore, $\lambda \neq 0$ because if $\lambda = 0$, then $z = 0$ which implies that $b = 0$. Hence by Lemma 3,

$$\begin{aligned}\int_0^{2\pi} &(a - b\cos t)^p \cos kt \,\mathrm{d}t \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (\lambda a - \lambda b \cos t)^p \cos kt \,\mathrm{d}t \\ &= \frac{1}{\lambda^p} \int_0^{2\pi} (z^2 + 1 - 2z\cos t)^p \cos kt \,\mathrm{d}t \\ &= \frac{(-1)^k \pi}{2^{p-1}} \sum_{q=0}^{p-k} \binom{p}{q} \binom{p}{k+q} C_{a,b,p,q,k}.\end{aligned}$$

□

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