Some new Fejér type inequalities via quantum calculus on finite intervals

Wengui Yang

School of Mathematics, Southeast University, Nanjing 210096, China
Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China
e-mail: wgyang0617@yahoo.com

ABSTRACT: Some new inequalities of Fejér type for twice differentiable mappings are established via quantum calculus on finite intervals. The results presented are extensions of those given earlier.

KEYWORDS: integral inequalities, differentiable mappings, Chebyshev quantum integral inequality

MSC2010: 26D10 34A08 26D15

INTRODUCTION

The function \( f : [a, b] \rightarrow \mathbb{R} \), is said to be convex if \( f_a(x, y) \leq af(x) + (1 - a)f(y) \) for all \( x, y \in [a, b] \), \( a \in [0, 1] \), where throughout the paper, \( F_q(x, y) \) denotes \( F(qx + (1 - q)y) \). We say that \( f \) is concave if \(-f\) is convex.

Let \( f : I \rightarrow \mathbb{R} \) be a convex mapping and \( a, b \in I \) with \( a < b \). The double inequality

\[
\frac{f(a + b)}{2} \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is well known as Hermite-Hadamard’s inequality for convex mapping\(^1,2\). Both inequalities hold in the reversed direction if \( f \) is concave. We note that (1) may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hermite-Hadamard’s inequality has received renewed attention in recent years due the fact that it has been considered the most useful inequality for convex functions in mathematical analysis\(^3\). For example, Xi and Qi\(^7\) and Chun and Qi\(^8\) obtained some Hermite-Hadamard’s inequalities for extended s-convex functions and functions whose third derivatives are convex, respectively. In Ref. 9 some Hermite-Hadamard type inequalities for \((p_1, h_1) - (p_2, h_2)\)-convex functions on the coordinates were established. We now state some known results.

**Theorem 1 (Ref. 10)** Let \( f : I \rightarrow \mathbb{R} \) be a convex mapping and \( a, b \in I \) with \( a < b \). Then

\[
f \left( \frac{a + b}{2} \right) \int_a^b w(x) \, dx \leq \left( \int_a^b f(x) w(x) \, dx \right) \leq \frac{f(a) + f(b)}{2} \int_a^b w(x) \, dx,
\]

where \( w : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable, and symmetric about \( x = (a + b)/2 \), i.e., \( w(x) = w(a + b - x) \).

**Theorem 2 (Refs. 11, 12)** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Then

\[
m \frac{(b - a)^2}{12} \leq f(a) + f(b) - \frac{1}{b - a} \int_a^b f(x) \, dx \leq M \frac{(b - a)^2}{12},
\]

and

\[
m \frac{(b - a)^2}{24} \leq \frac{1}{b - a} \int_a^b f(x) \, dx - f \left( \frac{a + b}{2} \right) \leq M \frac{(b - a)^2}{24}.
\]

**Theorem 3 (Ref. 13)** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Then, for
\( \lambda \in [0,1], \)
\[
\frac{\lambda(1-\lambda)}{2} (b-a)^2 \\
\leq \lambda f(a) + (1-\lambda)f(b) - f_\lambda(a,b) \\
\leq M \frac{\lambda(1-\lambda)}{2} (b-a)^2,
\]
and
\[
\frac{m(1-2\lambda)^2}{8} (b-a)^2 \\
\leq \frac{f_\lambda(f(a),b) + f_\lambda(b,a) - f\left(\frac{a+b}{2}\right)}{2} \\
\leq M \frac{(1-2\lambda)^2}{8} (b-a)^2.
\]

**Theorem 4 (Ref. 13)** Let \( f : [a,b] \to \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Assume that \( w : [a,b] \to \mathbb{R} \) is nonnegative, integrable, and symmetric about \( x = (a+b)/2 \). Then
\[
\frac{m}{2} \int_a^b (t-a)(b-a)w(t) dt \\
\leq \frac{f(a) + f(b)}{2} \int_a^b w(t) dt - \int_a^b f(t)w(t) dt \\
\leq M \frac{1}{2} \int_a^b (t-a)(b-a)w(t) dt,
\]
and
\[
\int_a^b (2t-a-b)^2 w(t) dt \\
\leq \int_a^b f(t)w(t) dt - f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \\
\leq M \int_a^b (2t-a-b)^2 w(t) dt.
\]

Tariboon and Ntouyas introduced quantum calculus on finite intervals in Ref. 14. Noor et al.\(^\text{15}\) applied a quantum analogue of the classical integral identity to establish some quantum estimates for Hermite-Hadamard inequalities for \( q \)-differentiable convex functions and \( q \)-differentiable quasi-convex functions. Tariboon and Ntouyas\(^\text{16}\) extended the Hőlder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovskoy-Schwarz, Grüss, and Grüss-Chebyshev integral inequalities to quantum calculus on finite intervals. Sudsutad et al.\(^\text{17}\) obtained some new Hermite-Hadamard type quantum integral inequalities for convex functions. Chen and Yang\(^\text{18}\) and Liu and Yang\(^\text{19}\) obtained some new Chebyshev and Grüss type inequalities via quantum calculus on finite intervals, respectively. Here we establish some new inequalities of Fejér type inequalities for differentiable mappings via quantum calculus on finite intervals. These are extensions of results given previously.

**PRELIMINARIES**

**Definition 1** [Ref. 14] Let \( I = [a,b] \subset \mathbb{R}, \quad t^0 = (a,b) \) and \( 0 < q < 1 \) be a constant. Assume \( f : I \to \mathbb{R} \) is a continuous function and let \( x \in I \). Then the expression
\[
aD_qf(x) = f(x) - f_q(x, a) \left(\frac{1-q}{x-a}\right), \quad x \neq a,
\]
\[
aD_qf(a) = \lim_{x \to a} aD_qf(x), \quad (6)
\]
is called the \( q \)-derivative on \( I \) of function \( f \) at \( x \).

We say that \( f \) is \( q \)-differentiable on \( I \) provided \( aD_qf(x) \) exists for all \( x \in I \). Note that if \( a = 0 \) in (6), then \( aD_qf = D_qf \), where \( D_q \) is the well-known \( q \)-derivative of the function \( f(x) \) defined by\(^\text{20}\)
\[
D_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}.
\]

**Definition 2** [Ref. 14] Assume \( f : I \to \mathbb{R} \) is a continuous function. Then the \( q \)-integral on \( I \) is defined by
\[
I_q^a f(x) = \int_a^x f(t) d_qt = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f_q^n(x, a), \quad (7)
\]
for \( x \in I \). If \( c \in (a,x) \) then the definite \( q \)-integral on \( I \) is defined by
\[
\int_c^x f(t) d_qt = \int_a^x f(t) d_qt - \int_a^c f(t) d_qt = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f_q^n(x, a)
\]
\[
\quad - (1-q)(c-a) \sum_{n=0}^{\infty} q^n f_q^n(c, a).
\]

Note that if \( a = 0 \), then (7) reduces to the classical \( q \)-integral of a function \( f(x) \) defined by\(^\text{20}\)
\[
\int_0^x f(t) d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad \forall x \in [0, \infty).
Lemma 1 (Ref. 18) Assume \( f, g : I \rightarrow \mathbb{R} \) are two continuous functions and \( f(t) \leq g(t) \) for all \( t \in I \). Then
\[
\int_a^x f(t) \, d_t t \leq \int_a^x g(t) \, d_t t.
\]

Theorem 5 (Ref. 14) Let \( f : I \rightarrow \mathbb{R} \) be a continuous function. Then we have
(i) \( \int_a^x f(t) \, d_t t = f(x); \)
(ii) \( \int_a^c f(t) \, d_t t = f(x) - f(c), \) for \( c \in (a, x). \)

Theorem 6 (Ref. 14) Let \( f, g : I \rightarrow \mathbb{R} \) be two continuous functions and \( a \in I \). Then, for \( x \in I \), we have
(i) \( \int_a^x [f(t) + g(t)] \, d_t t = \int_a^x f(t) \, d_t t + \int_a^x g(t) \, d_t t; \)
(ii) \( \int_a^x f(t) \, d_t t = \alpha \int_a^x f(t) \, d_t t; \)
(iii) \( \int_a^x f(t) \, d_t t = \int_a^x \lambda f(t) \, d_t t; \)
(iv) \( \int_a^x f(t) \, d_t t = \int_a^x (f(x)) a^x g(t) \, d_t t = [f g(t)] a^x t + \int_a^x f g(t) \, d_t t, \) for \( c \in (a, x). \)

FEJÉR TYPE INEQUALITIES VIA QUANTUM CALCULUS ON FINITE INTERVALS

Lemma 2 Let \( f : I \rightarrow \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Then, for \( \lambda \in [0, 1], \)
\[
m \lambda (1-\lambda) 2 \leq (1-\lambda) f(a) + \lambda f(b) - f_x(a, b)
\leq M \lambda (1-\lambda) 2 (b-a). \quad (8)
\]

Proof: The proof is done in the same way as that of Theorem 3.1 in Ref. 13.

Theorem 7 Let \( f : I \rightarrow \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Then
\[
\frac{Mq^2(b-a)^2}{(1+q)(1+q+q^2)} \leq \frac{Mq^2(b-a)^2}{(1+q)(1+q+q^2)} . \quad (9)
\]

and
\[
\frac{m(1+2q-2q^2+q^3)(b-a)^2}{4(1+q)(1+q+q^2)} \leq \frac{m(1+2q-2q^2+q^3)(b-a)^2}{4(1+q)(1+q+q^2)} . \quad (10)
\]

Proof: Integrating (8) with respect to \( \lambda \) over \([0, 1]\), we have
\[
m(b-a)^2 \int_0^1 \lambda (1-\lambda) 2 \, d_{\lambda} \lambda
\leq m(f(a)) + f(b) \int_0^1 \lambda 0 d_{\lambda} \lambda
\leq m f(b, a) \int_0^1 \lambda 0 d_{\lambda} \lambda
\leq M(b-a)^2 \int_0^1 \lambda (1-\lambda) 2 \, d_{\lambda} \lambda. \quad (11)
\]

By simple computation and changing the variable of (11), we obtain
\[
\frac{mq^2(b-a)^2}{2(1+q)(1+q+q^2)} \leq \frac{mq^2(b-a)^2}{2(1+q)(1+q+q^2)} . \quad (12)
\]

Similarly, integrating the first inequality of Theorem 3 with respect to \( \lambda \) over \([0, 1]\), we have
\[
m(b-a)^2 \int_0^1 \lambda (1-\lambda) 2 \, d_{\lambda} \lambda
\leq m(f(a)) + f(b) \int_0^1 \lambda 0 d_{\lambda} \lambda
\leq m f(b, a) \int_0^1 \lambda 0 d_{\lambda} \lambda
\leq M(b-a)^2 \int_0^1 \lambda (1-\lambda) 2 \, d_{\lambda} \lambda. \quad (13)
\]

By simple computation and changing the variable of (13), we obtain
\[
\frac{mq^2(b-a)^2}{2(1+q)(1+q+q^2)} \leq \frac{mq^2(b-a)^2}{2(1+q)(1+q+q^2)} . \quad (14)
\]

We obtain inequality (9) from (12) and (14). Integrating the second inequality of Theorem 3 with respect to \( \lambda \) over \([0, 1]\) and using the change of the variable, we obtain inequalities (10). 

www.scienceasia.org
Remark 1: If \( q^- \to 1 \), then inequalities (9) and (10) reduce to (2) and (3), respectively.

Theorem 8: Let \( f : I \to \mathbb{R} \) be a twice differentiable mapping such that there exist real constants \( m \) and \( M \) so that \( m \leq f'' \leq M \). Assume that \( w : I \to \mathbb{R} \) is nonnegative, integrable, and symmetric about \( x = (a + b)/2 \). Then

\[
m \int_a^b (x-a)(b-x)w(x) \, dx \\
\leq \int_a^b [f(a) + f(b)]w(x) \, dx \\
- \frac{1}{b-a} \int_a^b [f(x) + f(a+b-x)]w(x) \, dx \\
\leq M \int_a^b (x-a)(b-x)w(x) \, dx.
\]

Proof: Multiplying both sides of (8) by \( w_\lambda(b,a) \), and then integrating the resulting inequality with respect to \( \lambda \) over \([0,1]\), we have

\[
m(b-a)^2 \int_0^1 \frac{\lambda(1-\lambda)}{2} w_\lambda(b,a) \, d\lambda \\
\leq f(a) \int_0^1 (1-\lambda)w_\lambda(b,a) \, d\lambda \\
+ f(b) \int_0^1 \lambda w_\lambda(b,a) \, d\lambda \\
- \int_0^1 f_\lambda(b,a)w_\lambda(b,a) \, d\lambda \\
\leq M(b-a)^2 \int_0^1 \frac{\lambda(1-\lambda)}{2} w_\lambda(b,a) \, d\lambda.
\]

By simple computation and changing the variable of (17), we obtain

\[
m \int_a^b (x-a)(b-x)w(x) \, dx \\
\leq \frac{f(a)}{b-a} \int_a^b (b-x)w(x) \, dx \\
+ \frac{f(b)}{b-a} \int_a^b (x-a)w(x) \, dx \\
- \frac{1}{b-a} \int_a^b f(x)w(x) \, dx \\
\leq M \int_a^b (x-a)(b-x)w(x) \, dx.
\]

Similarly, multiplying both sides of the first inequality of Theorem 3 by \( w_\lambda(b,a) \), and then integrating the resulting inequality with respect to \( \lambda \) over \([0,1]\), we have

\[
m(b-a)^2 \int_0^1 \frac{\lambda(1-\lambda)}{2} w_\lambda(b,a) \, d\lambda \\
\leq f(a) \int_0^1 \lambda w_\lambda(b,a) \, d\lambda \\
+ f(b) \int_0^1 (1-\lambda)w_\lambda(b,a) \, d\lambda \\
- \int_0^1 f_\lambda(b,a)w_\lambda(b,a) \, d\lambda \\
\leq M(b-a)^2 \int_0^1 \frac{\lambda(1-\lambda)}{2} w_\lambda(b,a) \, d\lambda.
\]

By the symmetry of \( w(x) \) and changing the variable of (19), we obtain

\[
m \int_a^b (x-a)(b-x)w(x) \, dx \\
\leq \frac{f(a)}{b-a} \int_a^b (b-x)w(x) \, dx \\
+ \frac{f(b)}{b-a} \int_a^b (x-a)w(x) \, dx \\
- \frac{1}{b-a} \int_a^b f(a+b-x)w(x) \, dx \\
\leq M \int_a^b (x-a)(b-x)w(x) \, dx.
\]

We obtain (15) from (18) and (20). Multiplying both sides of the second inequality of Theorem 3 by
Proof: By integration by parts, we have that

\[
\int_a^b (x-a)(b-x)D_q^2 f(x)_q d_q x
\]

\[
= [(x-a)(b-x)]_a^b D_q f(x)_q d_q x
\]

\[
- \int_a^b [(a+b)-(1+q)x]_a D_q f_q(x,a)_a d_q x
\]

\[
= [(1+q)x-(a+b)]_a^b f_q(x,a)_q d_q x
\]

\[
- (1+q) \int_a^b f_q(x,a)_a d_q x
\]

\[
= (qb-a)f_q(b,a)+(b-qa)f(a)
\]

\[
- (1+q) \int_a^b f_q(x,a)_a d_q x
\]

which implies (22). Next, we prove (23). Using integration by parts, we obtain

\[
\int_a^b (x-a)^2 D_q^2 f(x)_q d_q x = [(x-a)^2 ]_a^b D_q f(x)_q d_q x
\]

\[
- \int_a^b [(1+q)x-2a]_a D_q f_q(x,a)_a d_q x
\]

\[
= (b-a)^2 D_q f(b)-[(1+q)x-2a]_a^b f_q(x,a)_q d_q x
\]

\[
+ (1+q) \int_a^b f_q(x,a)_a d_q x
\]

\[
= (b-a)^2 D_q f(b)-[(1+q)b-2a]_a^b f_q(b,a)
\]

\[
- (1-q)af(a)+(1+q) \int_a^b f_q(x,a)_a d_q x,
\]

and

\[
\int_a^b (x-b)^2 D_q^2 f(x)_q d_q x = [(x-b)^2 ]_a^b D_q f(x)_q d_q x
\]

\[
- \int_a^b [(1+q)x-2b]_a D_q f_q(x,a)_a d_q x
\]

\[
= -(b-a)^2 D_q f(b)
\]

\[
- [(1+q)x-2b]_a^b f_q(x,a)_q d_q x
\]

\[
+ (1+q) \int_a^b f_q(x,a)_a d_q x
\]

\[
= -(b-a)^2 D_q f(a)+(1-q)bf_q(b,a)
\]

\[
+ [(1+q)a-2b]_a^b f(a)
\]

\[
+ (1+q) \int_a^b f_q(x,a)_a d_q x.
\]

\[
\text{Remark 2 If } q \to 1, \text{ then inequalities (15) and (16) reduce to (4) and (5), respectively.}
\]

Lemma 3 Let \( f : I \to \mathbb{R} \) be twice \( q \)-differentiable with \( aD_q^2 f \) integrable on \( I \). Then

\[
\int_a^b (x-a)(b-x)D_q^2 f(x)_q d_q x
\]

\[
= (qb-a)f_q(b,a)+(b-qa)f(a)
\]

\[
- (1+q) \int_a^b f_q(x,a)_a d_q x,
\]

and

\[
\int_a^b [(x-a)^2+(x-b)^2]_a D_q^2 f(x)_q d_q x
\]

\[
= 2(1+q) \int_a^b f_q(x,a)_a d_q x-2(qb-a)f_q(b,a)
\]

\[
- 2(b-qa)f(a)+(b-a)^2 aD_q f(b)-aD_q f(a).
\]
Adding (24) and (25), we deduce (23).

Theorem 9 Let \( f : I \rightarrow \mathbb{R} \) be twice \( q \)-differentiable with \( aD_q^2f \) integrable on \( I \) and \( m \leq aD_q^2f \leq M \). Then

\[
\frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)} \leq (q-b)f_q(b,a) + (b-a)f(a) \\
-(1+q) \int_a^b f_q(x,a)_a d_q x \\
\leq \frac{Mq^2(b-a)^3}{(1+q)(1+q+q^2)},
\]

and

\[
\frac{m(1+q+q^3)(b-a)^3}{2(1+q)(1+q+q^2)} \\
\leq 2(1+q) \int_a^b f_q(x,a)_a d_q x \\
-2(q-b) a f_q(b,a) - 2(b-a) f(a) \\
+(b-a)^2(aD_q f(b) - aD_q f(a)) \\
\leq \frac{M(1+q+q^3)(b-a)^3}{2(1+q)(1+q+q^2)}. \tag{27}
\]

Proof: Since \( m \leq aD_q^2f \leq M \), we have

\[
m(x-a)(b-x) \leq (x-a)(b-x)_a D_q^2 f(x) \\
\leq M(x-a)(b-x), \quad \forall x \in [a, b]. \tag{28}
\]

Integrating (28) with respect to \( x \) from \( a \) to \( b \), we obtain

\[
m \int_a^b (x-a)(b-x)_a d_q x \\
\leq \int_a^b (x-a)(b-x)_a D_q^2 f(x)_a d_q x \\
\leq M \int_a^b (x-a)(b-x)_a d_q x. \tag{29}
\]

However, by (22), we have

\[
\int_a^b (x-a)(b-x)_a D_q^2 f(x)_a d_q x \\
= (q-b) f_q(b,a) + (b-a) f(a) \\
-(1+q) \int_a^b f_q(x,a)_a d_q x, \tag{30}
\]

We obtain (26) from (29)–(31).

From \( m \leq aD_q^2f \leq M \), we have

\[
m(x-a)^2 \leq (x-a)_a D_q^2 f(x) \leq M(x-a)^2, \tag{32}
\]

and

\[
m(x-b)^2 \leq (x-b)_a D_q^2 f(x) \leq M(x-b)^2, \tag{33}
\]

for \( x \in [a, b] \). Integrating (32) and (33) with respect to \( x \) from \( a \) to \( b \) we obtain

\[
m \int_a^b (x-a)^2 d_q x \leq \int_a^b (x-a)_a D_q^2 f(x)_a d_q x \\
\leq M \int_a^b (x-a)_a d_q x, \tag{34}
\]

and

\[
m \int_a^b (x-b)^2 d_q x \leq \int_a^b (x-b)_a D_q^2 f(x)_a d_q x \\
\leq M \int_a^b (x-b)_a d_q x, \tag{35}
\]

respectively. By simple computation, we have

\[
\int_a^b (x-a)^2 d_q x = \frac{(b-a)^3}{1+q+q^2}, \tag{36}
\]

\[
\int_a^b (x-b)^2 d_q x = \frac{(q+q^3)(b-a)^3}{(1+q)(1+q+q^2)}. \tag{37}
\]

We obtain (27) from Lemma 3 and (34)–(36).

Theorem 10 Let \( f : I \rightarrow \mathbb{R} \) be twice \( q \)-differentiable with \( aD_q^2f \) integrable on \( I \) and \( m \leq aD_q^2f \leq M \). Then

\[
\int_a^b (x-a)(b-x)_a d_q x = \frac{q^2(b-a)^3}{(1+q)(1+q+q^2)}. \tag{31}
\]
Proof: From Theorem 3.6 in Ref. 16, we obtain
\[
\left| \frac{1}{b-a} \int_a^b (x-a)(b-x)x D_q f(x) d_q x \right|
- \frac{1}{(b-a)^2} \int_a^b (x-a)(b-x) d_q x
\]
\[\times \int_a^b D_q^2 f(x) d_q x \leq \frac{1}{a} (K-k)(M-m), \tag{38}\]
where
\[
K = \sup_{x \in [a,b]} \{(x-a)(b-x)\} = \frac{(b-a)^2}{4},
\]
\[
k = \inf_{x \in [a,b]} \{(x-a)(b-x)\} = 0. \tag{39}\]

By Lemma 3, (31), (38) and (39), we have
\[
\left| \frac{1}{b-a} \left( (q-a)f_{q}(x,a) + (b-a)f_{q}(a) \right) \right|
- (1+q) \int_a^b f_{q}(x,a) d_q x
\]
\[\times \frac{Q(b-a)(b-a)D_q f(b) - aD_q f(a)}{(1+q)(1+q+q^2)} \leq \frac{(b-a)^2}{16} (M-m), \tag{40}\]
which implies (37). \[\square\]

Remark 3 If \(q \to 1\), then (37) reduces to the result obtained in Ref. 12.

Lemma 4 Let \(\varphi, g : I \to \mathbb{R}\) be two continuous functions and \(q\)-differentiable on \(I^0\). If \(aD_q g(x) \neq 0\) on \(I^0\) and \(m \leq aD_q \varphi(x) / aD_q g(x) \leq M\) on \(I^0\), then
\[
m \left( (b-a) \int_a^b g^2(x) d_q x - \left( \int_a^b g(x) d_q x \right)^2 \right)
\leq (b-a) \int_a^b \varphi(x)g(x) d_q x
\]
\[\left. - \int_a^b \varphi(x) d_q x \int_a^b g(x) d_q x \right) \leq M \left( (b-a) \int_a^b g^2(x) d_q x - \left( \int_a^b g(x) d_q x \right)^2 \right) \tag{41}\]

Proof: If \(aD_q g(x) \geq 0\), then \(g(x)\) is an increasing function and
\[
m aD_q g(x) \leq aD_q \varphi(x) \leq M aD_q g(x), \tag{42}\]
for \(\forall x \in I^0\). For \(a \leq x \leq y \leq b\), integrating (41) with respect to \(x\) from \(x\) to \(y\), we obtain
\[
m(g(y) - g(x)) \leq \varphi(y) - \varphi(x) \leq M(g(y) - g(x)). \tag{43}\]
Multiplying both sides of (42) by \(g(y) - g(x) \geq 0\), then
\[
m(g(y) - g(x))^2 \leq (\varphi(y) - \varphi(x))(g(y) - g(x)) \leq M(g(y) - g(x))^2. \tag{44}\]
Similarly, if \(aD_q g(x) \leq 0\), we can also obtain (43). Integrating the obtained result (43) with respect to \(x\) and \(y\) from \(a\) to \(b\), we have
\[
m \int_a^b \int_a^b (g(y) - g(x))^2 d_q x d_q y
\leq \int_a^b \int_a^b (\varphi(y) - \varphi(x))(g(y) - g(x)) d_q x d_q y
\leq M \int_a^b \int_a^b (g(y) - g(x))^2 d_q x d_q y. \tag{45}\]
A simple calculation shows us that
\[
\int_a^b \int_a^b (g(y) - g(x))^2 d_q x d_q y
\leq \left( \int_a^b g(x) d_q x \right)^2 \tag{46}\]
and
\[
\int_a^b \int_a^b (\varphi(y) - \varphi(x))(g(y) - g(x)) d_q x d_q y
\leq \left( \int_a^b \varphi(x) d_q x \right)^2 \tag{47}\]
We obtain (40) from (44)–(46). \[\square\]

Lemma 5 Let \(f : I \to \mathbb{R}\) be twice \(q\)-differentiable with \(aD_q f\) integrable on \(I\). Then
\[
\int_a^b (2x-a-b) aD_q f(x) d_q x
= (b-a)(f(a) + f(b)) - 2 \int_a^b f_q(x, a) d_q x. \tag{48}\]
Lemma 3. Proof with Theorem 11

Let $f: I \to \mathbb{R}$ be twice $q$-differentiable with $D_q^2 f$ integrable on $I$ and $m \leq D_q^2 f \leq M$. Then

$$mq(b-a)^3 (1+q)^2 (1+q+q^2) \leq (b-a)((1+q)f(a)+2qf(b)) \leq 2(1+q)$$

Proof: Method 1. From Lemma 4, we choose $\phi(x) = D_q f(x)$ and $g(x) = x - \frac{1}{2} (a+b)$. Then $m \leq D_q \phi(x)/D_q g(x) \leq M$ holds on $(a,b)$. By Lemma 4,

$$m \left( (b-a) \int_a^b \left( x - \frac{a+b}{2} \right)^2 D_q x \right) \leq (b-a) \int_a^b \left( x - \frac{a+b}{2} \right) D_q f(x) D_q x \leq m \left( (b-a) \int_a^b \left( x - \frac{a+b}{2} \right)^2 D_q x \right)$$

(49)

A simple calculation shows that

$$\int_a^b y(x) D_q x = \frac{(1-q)(b-a)^2}{2(1+q)},$$

$$\int_a^b y(x)^2 D_q x = \frac{1+2q-2q^2+q^3}{4(1+q)(1+q+q^2)},$$

(50)

where $y(x) = x+(a+b)/2$, and

$$\int_a^b \left( x - \frac{a+b}{2} \right) D_q f(x) D_q x = \frac{(b-a)(f(a)+f(b))}{2} - \int_a^b f(x, a)_D D_q x. \quad (51)$$

We obtain (48) from (49)–(51).

Method 2. The identity (47) is expressed as

$$(b-a)(f(a)+f(b)) - 2 \int_a^b f(x, a)_D D_q f(x) D_q x$$

$$= \int_a^b f(x, a)_D D_q f(x) D_q x + \int_a^b D_q f(x, a)_D D_q x. \quad (52)$$

Now we apply Lemma 4 to obtain

$$m \left( \int_a^b (x-a)^2 D_q x - \Lambda_a \right)$$

$$\leq (b-a) \int_a^b (b-a)_D D_q f(x) D_q x$$

$$- \Lambda_a \int_a^b D_q f(x) D_q x$$

$$\leq M \left( (b-a) \int_a^b (x-a)^2 D_q x - \Lambda_a^2 \right). \quad (53)$$

where $\Lambda_a = \int_a^b (x-a)_D D_q x$. A simple calculation and (ii) of Theorem 3.2 in Ref. 14 show that

$$\int_a^b (x-a)_D D_q x = \frac{(b-a)^2}{1+q},$$

$$\int_a^b (x-a)^2 D_q x = \frac{(b-a)^3}{1+q+q^2},$$

(54)

From (53) and (54), we obtain

$$\frac{mq(b-a)^3}{(1+q)^2 (1+q+q^2)} \leq (b-a) \int_a^b (b-a)_D D_q f(x) D_q x$$

$$- \frac{(b-a)^2}{1+q} f(b)$$

$$\leq \frac{Mq(b-a)^3}{(1+q)^2 (1+q+q^2)},$$

which implies

$$\frac{mq(b-a)^3}{(1+q)^2 (1+q+q^2)} \leq \int_a^b (x-a)_D D_q f(x) D_q x - \frac{b-a}{1+q} f(b)$$

$$\leq \frac{Mq(b-a)^3}{(1+q)^2 (1+q+q^2)}. \quad (55)$$
Similarly, Lemma 4 gives

\[ m \left( (b-a) \int_a^b (x-b)^2 d_q x - \Lambda_b^2 \right) \]
\[ \leq (b-a) \int_a^b (x-b) D_q f(x) d_q x \]
\[ - \Lambda_b \int_a^b D_q f(x) d_q x \]
\[ \leq M \left( (b-a) \int_a^b (x-b)^2 d_q x - \Lambda_b^2 \right), \quad (56) \]

where \( \Lambda_b = \int_a^b (x-b) d_q x \). A simple calculation shows that

\[ \int_a^b (x-b) d_q x = - \frac{q(b-a)^2}{1+q}, \]
\[ \int_a^b (x-b)^2 d_q x = \frac{(q+3)(b-a)^3}{(1+q)(1+q+q^2)}. \quad (57) \]

From (56) and (57), we obtain

\[ \frac{mq(b-a)^4}{(1+q)^2(1+q+q^2)} \]
\[ \leq (b-a) \int_a^b (x-b) D_q f(x) d_q x \]
\[ + \frac{q(b-a)^2}{1+q} f(b) \]
\[ \leq \frac{Mq(b-a)^4}{(1+q)^2(1+q+q^2)}, \]

which implies

\[ \frac{mq(b-a)^3}{(1+q)^2(1+q+q^2)} \]
\[ \leq \int_a^b (x-b) D_q f(x) d_q x + \frac{q(b-a)^2}{1+q} f(b) \]
\[ \leq \frac{Mq(b-a)^3}{(1+q)^2(1+q+q^2)}. \quad (58) \]

We now add (55) and (58), taking into account the identity (52). Then we obtain (48) from Lemma 5.

\[ \square \]

Remark 4 From Theorem 11, we obtain the same result from two different methods.

Theorem 12 Let \( f : I \to \mathbb{R} \) be twice \( q \)-differentiable with \( a D_q^2 f \) integrable on \( I \) and \( m \leq a D_q^2 f \leq M \). Then

\[ \frac{Mq(b-a)^3}{(1+q)^2(1+q+q^2)} + B \]
\[ \leq (b-a) f_q(b, a) + (b-qa)f(a) \]
\[ - (1+q) \int_a^b f_q(x, a) d_q x \]
\[ \leq A + \frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)}, \quad (59) \]

where

\[ A = \left[ (b-a) \left( a D_q f(b) - \frac{m(b-a)}{1+q} \right) \right. \]
\[ - \frac{f_q(b, a) - f(a)}{\left( a D_q f(b) - a D_q f(a) - m(b-a) \right)} \]
\[ \times \left\{ (b-a)(a D_q f(a) - \frac{mq(b-a)}{1+q}) \right. \]
\[ - \frac{a D_q f(b) - a D_q f(a) - m(b-a)}{a D_q f(b) - a D_q f(a) - m(b-a)} \right\}, \]

and

\[ B = \left[ (b-a) \left( \frac{M(b-a)}{1+q} - a D_q f(b) \right) \right. \]
\[ + \left( f_q(b, a) - f(a) \right) \]
\[ \times \left[ \left( \frac{\int_a^b f_q(x, a) d_q x}{M(b-a) - (a D_q f(b) - a D_q f(a))} \right) \right. \]
\[ - \left( \frac{Mq(b-a)}{1+q} + a D_q f(a) \right) \]
\[ \left. \times \left( \frac{Mq(b-a)}{1+q} + a D_q f(a) \right) \right\}, \]

provided that \( a D_q f(b) - a D_q f(a) \neq m(b-a) \) and \( a D_q f(b) - a D_q f(a) \neq M(b-a) \).

Proof: To prove (59), we firstly give the following weighted Chebyshev type quantum integral inequality on finite intervals:

\[ \int_a^b \phi(x) d_q x \int_a^b \phi(x) g(x) h(x) d_q x \]
\[ \leq \int_a^b \phi(x) g(x) d_q x \int_a^b \phi(x) h(x) d_q x, \quad (60) \]

where \( g \) and \( h \) are continuous asynchronous functions and \( \phi \) is a continuous nonnegative function. Two functions \( g \) and \( h \) are said to be asynchronous on \( [a, b] \), if \( (g(x) - g(y))(h(x) - h(y)) \leq 0 \), for
any $x, y \in [a, b]$. In fact, from Definition 2 and following the proof of Lemma 3.1 in Ref. 16, we have

\[
\int_a^b \int_a^b \phi(x)\phi(y)(g(y) - g(x))(h(y) - h(x))_a d_q x \, d_q y \\
= 2 \left( \int_a^b \phi(x)_a d_q x \int_a^b \phi(x)g(x)h(x)_a d_q x \\
- \int_a^b \phi(x)g(x)_a d_q x \int_a^b \phi(x)h(x)_a d_q x \right). \tag{61}
\]

Since $g$ and $h$ are continuous asynchronous functions and $\phi$ is continuous nonnegative function, we have

\[
\phi(x)\phi(y)(g(y) - g(x))(h(y) - h(x)) \leq 0. \tag{62}
\]

From (61) and (62), we obtain (60).

Now let

\[
\Omega = \int_a^b (x-a)(b-x)_a D_q^2 f(x)_a d_q x \\
= (q b - a)f_q(b, a) + (b - qa)f(a) \\
- (1 + q) \int_a^b f_q(x, a)_a d_q x.
\]

From Lemma 3 we can easily obtain

\[
\int_a^b (x-a)(b-x)(a D_q^2 f(x) - m)_a d_q x \\
= \Omega - m \int_a^b (x-a)(b-x)_a d_q x \\
= \Omega - \frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)}, \tag{63}
\]

and

\[
\int_a^b (x-a)(b-x)(M - a D_q^2 f(x))_a d_q x \\
= M \int_a^b (x-a)(b-x)_a d_q x - \Omega \\
= \frac{Mq^2(b-a)^3}{(1+q)(1+q+q^2)} - \Omega. \tag{64}
\]

From (60), we obtain

\[
\int_a^b (x-a)(b-x)(a D_q^2 f(x) - m)_a d_q x \\
\leq \int_a^b (x-a)(b-x)(a D_q^2 f(x) - m)_a d_q x \\
\leq \int_a^b (b-x)(b D_q^2 f(x) - m)_a d_q x \\
\times \frac{\int_a^b (a D_q^2 f(x) - m)_a d_q x}{\int_a^b (b D_q^2 f(x) - m)_a d_q x}, \tag{65}
\]

and

\[
\int_a^b (x-a)(b-x)(M - a D_q^2 f(x))_a d_q x \\
\leq \int_a^b (x-a)(M - a D_q^2 f(x))_a d_q x \\
\leq \int_a^b (b-x)(M - a D_q^2 f(x))_a d_q x \\
\times \frac{\int_a^b (M - a D_q^2 f(x))_a d_q x}{\int_a^b (M - a D_q^2 f(x))_a d_q x}. \tag{66}
\]

A simple calculation shows us that

\[
\int_a^b \int_a^b (a D_q^2 f(x) - m)_a d_q x \\
= \int_a^b a D_q (a D_q f(x) - mx)_a d_q x \\
= (a D_q f(b) - a D_q f(a) - m(b-a)), \tag{67}
\]

\[
\int_a^b (x-a)(a D_q^2 f(x) - m)_a d_q x \\
= \int_a^b (x-a) a D_q (a D_q f(x) - mx)_a d_q x \\
= (x-a)(a D_q f(x) - mx)|_a^b \\
- \int_a^b (a D_q f(x + (1-q)x) - m(qx, a))_a d_q x \\
= (b-a)(a D_q f(b) - mb) \\
- \left( f_q(b, a) - f(a) - \frac{m(b-a)(bq+a)}{1+q} \right) \\
= (b-a)\left( a D_q f(b) \\
- \frac{m(b-a)}{1+q} \right) - (f_q(b, a) - f(a)), \tag{68}
\]

www.scienceasia.org
and
\[
\int_a^b (b-x)[(b-a)D_q^2f(x) - m]_a d_q x
\]
\[
= \int_a^b (b-x)[(b-a)D_q f(x) - mx]_a d_q x
\]
\[
= (b-a)[D_q f(a) - ma]
\]
\[
+ \left(f_q(b,a) - f(a) - \frac{m(b-a)(bq+a)}{1+q}\right)
\]
\[
= (f_q(b,a) - f(a))
\]
\[
- (b-a)\left(D_q f(a) - \frac{mq(b-a)}{1+q}\right).
\]  
(69)

From (63), (65), (67)–(69), we have
\[
\Omega - \frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)} \leq \left[(b-a)[(b-a)D_q f(b) - \frac{m(b-a)}{1+q} - \left(f_q(b,a) - f(a)\right)]
\]
\[
\times \left(f_q(b,a) - f(a) - \frac{m(b-a)(bq+a)}{1+q}\right)
\]
\[
= \left(1 - \frac{M(b-a)}{1+q}\right)\left(1 - D_q f(b)\right) - \left(D_q f(a) - \frac{mq(b-a)}{1+q}\right).
\]  
(70)

Similarly, we have
\[
\int_a^b \int_a^b (M-a)D_q^2 f(x)_a d_q x
\]
\[
= \int_a^b aD_q(Mx - aD_q f(x))_a d_q x
\]
\[
= M(b-a) - (M-a)D_q f(b).
\]  
(71)

\[
\int_a^b (x-a)(M-a)D_q^2 f(x)_a d_q x
\]
\[
= \int_a^b (x-a)aD_q(Mx - aD_q f(x))_a d_q x
\]
\[
= (x-a)(Mx - aD_q f(x))_a
\]
\[
- \int_a^b (Mx + (1-q)a) - aD_q f(x, a)_a d_q x
\]
\[
= (b-a)(Mb - aD_q f(b)) - \left(M(b-a)(bq+a)\right)
\]
\[
- \left(f_q(b,a) - f(a)\right),
\]  
(72)

and
\[
\int_a^b (b-x)(M-a)D_q^2 f(x)_a d_q x
\]
\[
= \int_a^b (b-x)aD_q(Mx - aD_q f(x))_a d_q x
\]
\[
= (b-x)(Mx - aD_q f(x))_a
\]
\[
+ \int_a^b (Mx + (1-q)a) - aD_q f(x, a)_a d_q x
\]
\[
= -(b-a)(Ma - aD_q f(a))
\]
\[
+ \frac{M(b-a)(bq+a)}{1+q} - f_q(b,a) + f(a)
\]
\[
= (b-a)(\frac{Mq(b-a)}{1+q} + aD_q f(a)) - f_q(b,a) + f(a).
\]  
(73)

From (64), (66), (71)–(73), we have
\[
\frac{Mq^2(b-a)^3}{(1+q)(1+q+q^2)} - \Omega
\]
\[
\leq \left[(b-a)[M(b-a) 1+q - aD_q f(b)]
\]
\[
+ f_q(b,a) - f(a)\right]
\]
\[
\times \left(\frac{Mq(b-a)}{1+q} + aD_q f(a)\right)
\]
\[
= \left(\frac{M(b-a)}{1+q} - D_q f(b)\right)\left(\frac{M(b-a)}{1+q} - aD_q f(b)\right)
\]
\[
- \left(D_q f(a) - \frac{mq(b-a)}{1+q}\right).
\]  
(74)

We now obtain (59) from (70) and (74).

**Remark 5** If \( q \to 1 \), then Lemma 4–Theorem 12 reduce to the results obtained in Ref. 2.

**Remark 6** Let \( f(x) = x^2 \in [a, b] \). It is easy to see that \( f : I \to \mathbb{R} \) is a twice differentiable mapping such that \( f'' = 2 \). From Theorem 7, we have
\[
\frac{2q^2(b-a)^2}{(1+q)(1+q+q^2)}
\]
\[
\leq \left(a^2 + b^2\right)\left(\frac{2q^2 + q^3}{1+q}\right) + 4q^2 \frac{ab}{(1+q)(1+q+q^2)}
\]
\[
\leq \frac{2q^2(b-a)^2}{(1+q)(1+q+q^2)}.
\]  
(75)
and
\[
\frac{(1 + 2q - 2q^2 + q^3)(b - a)^2}{2(1 + q)(1 + q + q^2)} \leq \frac{(a^2 + b^2)(1 + 2q + q^3) + 4q^2ab}{(1 + q)(1 + q + q^2)} - 2\left(\frac{a + b}{2}\right)^3 \\
\leq \frac{(1 + 2q - 2q^2 + q^3)(b - a)^2}{2(1 + q)(1 + q + q^2)}. 
\]

(76)

It is easy to see that inequalities (75) and (76) reduce to two equalities.

**Remark 7** Let \( f(x) = x^3 \in [a, b] \). Then \( (3 - q + 4q^2)a = m \leq \frac{1}{2}D_2^2f \leq M = (1 - q)(2 - q + q^2)a + (1 + q)(1 + q + q^2)b \). From Theorem 11, we have
\[
\frac{mq(b - a)^2}{(1 + q)^2(1 + q + q^2)} \leq \frac{(1 + q)a^3 + 2q^3}{2(1 + q)} \\
- \frac{\left(\frac{q^2(b - a)}{1 + q + q^2 + q^3} + \frac{3q^2a(b - a)^2}{1 + q + q^2} + 3q^2a^2(b - a)^2}{1 + q + q^2}
\]
\[
+ \frac{3q^2a^2(b - a)^2}{1 + q} + a^3 \leq \frac{Mq(b - a)^2}{(1 + q)^2(1 + q + q^2)}. 
\]

**REFERENCES**