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Approximate solution of space-time fractional differential equations via Legendre polynomials

K. Krishnaveni*, **K. Kannan**, **S. Raja Balachandar**, **S.G. Venkatesh**

Department of Mathematics, School of Humanities and Sciences, SASTRA University, Thanjavur-613401, Tamilnadu, India

*Corresponding author, e-mail: drkveni2012@gmail.com

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ABSTRACT: In this paper, we present a new method based on the fractional shifted Legendre polynomials to solve non-homogeneous space and time fractional partial differential equations (FPDEs) in which space and time fractional derivatives are described in the Caputo sense. The main purpose of this technique is to transform the FPDE into algebraic equations which can be solved easily. Convergence and error analysis show the correctness and feasibility of the method. The applicability and efficiency of the proposed approach are illustrated by some examples.

KEYWORDS: shifted Legendre polynomial, Caputo fractional derivative, convergence analysis

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INTRODUCTION

Fractional calculus has been investigated by many authors[1,2]. Recently, Momani used an Adomian decomposition method[3] to find the non-perturbative analytical solutions of the fractional Burgers' equations and Inc[4] used the variational iteration method to solve the same equation. Homotopy perturbation and Adomian decomposition methods have been used by Wang[5] to construct approximate solutions of the nonlinear fractional KdV-Burgers' equation. Al-Khaled and Momani[6] proposed the decomposition method to obtain an approximate solution of the generalized time-fractional diffusion-wave equation. Mainardi[7] considered analytical investigation of the time-fractional diffusion wave equations and provided a comprehensive review of research on the application of calculus in continuum and statistical mechanics including research on fractional diffusion-wave solutions. This brief review of fractional diffusion-wave equations and their applications is by no means complete. References to other papers akin to fractional diffusion-wave equations can be found in Ref. 7. Kumar[8,9] proposed modified Laplace transform and fractional homotopy analysis transform methods for solving a time-fractional Fokker-Planck equation and nonlinear homogeneous and non-homogeneous time-fractional gas dynamics equations, respectively.

A space-time fractional diffusion equation can be used as a linear model for a physical system involving a linear diffusion equation. Here, we consider the following non-homogeneous space-time fractional differential equation[10]:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^\beta u(x,t)}{\partial t^\beta} + f(u,x,t),$$
$$1 < \alpha \leqslant 2, \quad 0 < \beta \leqslant 1,$$

subject to the boundary and initial conditions

$$u(0,t) = h_1(t),\ u(a,t) = h_2(t),\ t \geqslant 0,$$
$$u(x,0) = f(x),\ 0 < x < l,$$
$$\frac{\partial u(x,0)}{\partial x} = g(t),\ 0 < x < l,$$

where α and β are parameters describing the order of space and time fractional derivatives, respectively, and $u(x,t)$ is the field defined in the space domain $[0,l]$. The main objective of the present article lies in introducing a new analytical or approximate solution of time-space fractional diffusion wave equations by means of fractional order shifted Legendre polynomials. The suggested method is a potential tool for transforming fractional partial differential equations (FPDEs) to algebraic equations with fractional initial value conditions.

PRELIMINARIES AND NOTATION

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and integration[11]. Various definitions of fractional integration and differentiation are available[2,12]. The Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional-order initial conditions. Caputo introduced an alternative definition which has the advantage of defining integer-order initial conditions for fractional-order differential equations (FDEs)[2].

A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty]$. Clearly, $C_\mu < C_\beta$ if $\beta<\mu$. A function $f(x)$, $x>0$, is said to be in the space C_μ^m, $m\in\mathbb{N}\cup\{0\}$ if $f^{(m)}\in C_\mu$. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} D_*^\alpha f(x) &= J^{m-\alpha}D^m f(x)\\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,\mathrm{d}t, \end{aligned}$$

for $m-1<\alpha\leqslant m, m\in\mathbb{N}, x>0, f\in C_{-1}^m$. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Note that $D^\alpha C=0$ where C is a constant and

$$D^\alpha D^\beta f(x) = D^{\alpha+\beta} f(x),$$

$$D^\alpha x^\beta = \begin{cases} 0, & \beta\in\mathbb{N}_0,\ \beta<\lceil\alpha\rceil,\\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \beta\in\mathbb{N}_0,\ \beta\geqslant\lceil\alpha\rceil,\\ & \beta\notin N,\ \beta>\lfloor\alpha\rfloor. \end{cases}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α and $\lfloor\alpha\rfloor$ is the largest integer less than or equal to α. Also $\mathbb{N}=\{1,2,\dots\}$ and $\mathbb{N}_0=\{0,1,2,\dots\}$. As with the integer-order derivative, the Caputo fractional derivative is a linear operation: $D^\alpha(\sum_{i=1}^n c_i f_i(t)) = \sum_{i=1}^n c_i D^\alpha f_i(t)$, where $\{c_i\}_{i=1}^n$ are constants.

FRACTIONAL-ORDER LEGENDRE FUNCTIONS

In this section, we discuss the properties of fractional order Legendre functions and the function approximation based on shifted Legendre polynomials and their properties.

Shifted Legendre polynomials and their properties

The Legendre polynomials[13] $P_n(z)$ satisfy

$$\int_{-1}^1 P_n(z)P_m(z)\,\mathrm{d}z = \frac{2}{2n+1}\delta_{nm}.$$

For use on the interval $t\in(0,1)$ we define the so-called shifted Legendre polynomials by introducing the change of variable $z=2t-1$. Let the shifted Legendre polynomials $P_n(2t-1)$ be denoted by $L_n(t)$. Then

$$\int_0^1 L_n(t)L_m(t)\,\mathrm{d}t = \frac{\delta_{nm}}{2n+1}.$$

Then for $i=1,2,\dots$

$$L_{i+1}(t) = \frac{(2i+1)(2t-1)}{i+1}L_i(t) - \frac{i}{i+1}L_{i-1}(t), \quad (1)$$

where $L_0(t)=1$ and $L_1(t)=2t-1$. In general, $L_n(t)=\sum_{i=0}^n(-1)^{n+i}((n+i)!/(n-i)!)t^i/(i!)^2$. Note that $L_n(0)=(-1)^n$ and $L_n(1)=1$.

Fractional-order Legendre definition for FPDE

The Adomian decomposition method[14], the homotopy perturbation method[15] and He's variational iteration method[16] are the commonly used methods to solve FDEs of order α which involve the series expansion of the form $\sum_{i=0}^N c_i x^{i\alpha}$. Thus construction of the orthogonal functions of the form $\Phi_n(x)=\sum_{i=0}^n c_i x^{i\alpha}$ help us to solve FDEs more efficiently. Recently, Rida and Yousef[17] have generated a fractional extension of the classical Legendre polynomials by replacing the integer order derivative in Rodrigues' formula by fractional order derivatives. But the complexity of these functions make them difficult for solving FDEs. Subsequently, Kazem[18] generated the orthogonal fractional order Legendre functions based on shifted Legendre polynomials to obtain the solution of FDEs more simply and efficiently.

The fractional-order Legendre functions (FLFs) introduced by Kazem[18], are defined by introducing the change of variable $t=x^\alpha$ and $\alpha>0$ on the shifted Legendre polynomials. Let the FLFs $L_i(x^\alpha)$ be denoted by $\mathscr{L}_i^\alpha(x)$. The fractional-order Legendre functions are a particular solution of the normalized eigenfunctions of the singular Sturm-Liouville problem

$$((x-x^{1+\alpha})\mathscr{L}_i^{\prime\alpha}(x))' + \alpha^2 i(i+1)x^{\alpha-1}\mathscr{L}_i^\alpha(x) = 0,$$

for $x\in(0,1)$. Then $\mathscr{L}_i^\alpha(x)$ by using (1) can be obtained from

$$\mathscr{L}_i^\alpha(x) = \frac{(2i+1)(2x^\alpha-1)}{i+1}\mathscr{L}_i^\alpha(x) - \frac{i}{i+1}\mathscr{L}_{i-1}^\alpha(x),$$

for $i=1,2,\dots$, where $\mathscr{L}_0^\alpha(x)=1$ and $\mathscr{L}_1^\alpha(x)=2x^\alpha-1$. Generally, $\mathscr{L}_i^\alpha(x)=\sum_{s=0}^i b_{s,i}x^{s\alpha}$, where

$b_{s,i} = (-1)^{i+s}(i+s)!/(i-s)!(s!)^2$. Note that $\mathscr{L}_i^\alpha(0) = (-1)^i$ and $\mathscr{L}_i^\alpha(1) = 1$. The FLFs obey

$$\int_0^1 \mathscr{L}_n^\alpha(x)\mathscr{L}_m^\alpha(x)w(x)\,\mathrm{d}x = \frac{\alpha\delta_{nm}}{2n+1}.$$

For solving FPDEs, we define

$$u(x,t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} c_{ij}\mathscr{L}_i^\alpha(t)\right)\phi_j(x^\alpha) \qquad (2)$$

$$D_x^\gamma u(x,t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} c_{ij}D^\gamma\mathscr{L}_i^\alpha(t)\right)\phi_j(x^\alpha)$$

$$D_t^\lambda u(x,t) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} c_{ij}\mathscr{L}_i^\alpha(t)\right)\frac{\partial^\lambda}{\partial x^\lambda}\phi_j(x^\alpha)$$

where

$$D^\gamma\mathscr{L}_i^\alpha(t) = \sum_{s=0}^{i} b'_{s,i}\frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)}t^{s\alpha-\gamma},$$

$$\phi_j(x^\alpha) = x^{j\alpha}$$

and

$$c_{ij} = \alpha(2i+1)(j+1)\int_0^1\int_0^1 G(x,t,\alpha)\,\mathrm{d}t\,\mathrm{d}x,$$

where $G(x,t,\alpha) = \mathscr{L}_i^\alpha(t)w(t)w(x)u(x,t)\phi_j(x^\alpha)$.

Function approximation

In practice, only the first nm terms of (2) are considered. Then we have

$$u_{m,n}(x,t) = \sum_{j=0}^{n-1}\left(\sum_{i=0}^{m-1} c_{ij}\mathscr{L}_i^\alpha(t)\right)\phi_j(x^\alpha) = C^{\mathrm{T}}\mathscr{L}^\alpha(t)\Phi(x^\alpha), \qquad (3)$$

where C and $\mathscr{L}^\alpha(t)$ are given by

$C = [c_{00}, c_{10}, c_{20}, \dots, c_{m-1\ 0}, c_{01}, c_{11}, c_{21}, \dots, c_{m-1\ 1}, \dots, c_{0\ n-1}, c_{1\ n-1}c_{2\ n-1}, \dots, c_{m-1\ n-1}]^{\mathrm{T}}$,
$\mathscr{L}^\alpha(t) = [\mathscr{L}_0^\alpha(t), \mathscr{L}_1^\alpha(t), \dots, \mathscr{L}_{m-1}^\alpha(t)]^{\mathrm{T}}$,
$\Phi(x^\alpha) = [\phi_0(x^\alpha), \phi_1(x^\alpha), \dots, \phi_{n-1}(x^\alpha)]^{\mathrm{T}} = [1, x^\alpha, x^{2\alpha}, \dots, x^{\alpha(n-1)}]^{\mathrm{T}}$,

and

$$c_{ij} = \alpha(2i+1)(j+1)\int_0^1\int_0^1 u(x,t)\mathscr{L}_i^\alpha(t)\, x^{\alpha-1}t^{\alpha-1}\phi_j(x^\alpha)\,\mathrm{d}t\,\mathrm{d}x.$$

Description of the method

First, we re-express the unknown approximation functions:

$$u_{m,n}(x,t) = \sum_{j=0}^{n-1}\left(\sum_{i=0}^{m-1} c_{ij}\mathscr{L}_i^\beta(t)\right)\Phi_j(x^\alpha),$$

where $\Phi_j(x^\alpha) = x^{j\alpha}$.

$$D_t^\beta u_{m,n}(x,t) = \sum_{j=0}^{n-1}\sum_{i=0}^{m-1} c_{ij}\frac{\partial^\beta}{\partial t^\beta}(\mathscr{L}_i^\beta(t))\Phi_j(x^\alpha).$$

$$D_x^\alpha u_{m,n}(x,t) = \sum_{j=o}^{n-1}\left(\sum_{i=0}^{m-1} c_{ij}\mathscr{L}_i^\beta(t)\right)\frac{\partial^\alpha}{\partial x^\alpha}\Phi_j(x^\alpha).$$

The number of unknown coefficients c_{ij} is equal to nm and it can be obtained from the equations and initial boundary conditions to solve any space-time FPDEs. If the exact solution to the problem is known, the accuracy and efficiency of the proposed method based on maximum absolute error $e_{m,n}$ for $a \leqslant x \leqslant b$, and $0 < t < \tau$ is defined as

$$e_{m,n} = \max\{|u(x,t) - u_{m,n}(x,t)|\}.$$

THEORETICAL ANALYSIS

In this section, we discuss the convergence analysis and error estimation for the proposed technique.

Theorem 1 *A continuous function $u(x,t)$ with second derivative bounded by M (a constant), can be expressed as in* (2) *and the truncated series given in* (3) *converges towards the exact solution of the FPDE.*

Proof: Consider

$$a_{ij} = \alpha(2i+1)(j+1)\int_0^1 u_{m,n}(x,t)\mathscr{L}_i^\alpha(t)\ t^{\alpha-1}\,\mathrm{d}t.$$

Here $\mathscr{L}_i^\alpha(t) = P_i(2t^\alpha - 1)$, where P_i are Legendre polynomials in x and the fractional shifted Legendre polynomials are orthogonal with respect to the weight function $w(t) = x^{\alpha-1}t^{\alpha-1}$ in the interval $(0,1]$.

$$a_{ij} = \alpha(2i+1)(j+1)\int_0^1 u_{m,n}(x,t)P_i(2t^\alpha-1)t^{\alpha-1}\mathrm{d}t$$
$$= \frac{(2i+1)(j+1)}{2}\int_{-1}^1 u_{m,n}(x,t)P_i(v)\,\mathrm{d}v$$
$$= \frac{j+1}{2}\int_{-1}^1 u_{m,n}(x,t)\,\mathrm{d}[P_{i+1}(v) - P_{i-1}(v)]$$

$$= -\frac{j+1}{2}\int_{-1}^{1} u'_{m,n}(x,t)[P_{i+1}(v)-P_{i-1}(v)]\,\mathrm{d}v$$
$$= -\frac{j+1}{2}\int_{-1}^{1} u'_{m,n}(x,t)$$
$$\mathrm{d}\left[\frac{P_{i+2}(v)-P_i(v)}{2i+3}-\frac{P_i(v)-P_{i-2}(v)}{2i-1}\right]$$
$$= \tfrac{1}{2}(j+1)\int_{-1}^{1} u''_{m,n}(x,t)$$
$$\left[\frac{P_{i+2}(v)-P_i(v)}{2i+3}-\frac{P_i(v)-P_{i-2}(v)}{2i-1}\right]\mathrm{d}v.$$

Consider

$$\left|\frac{1}{2}\int_{-1}^{1}(j+1)u''_{m,n}(x,t)\,\mathrm{d}vB_1\right|^2$$
$$= \left|\frac{1}{2}\int_{-1}^{1}(j+1)u''_{m,n}(x,t)B_1\,\mathrm{d}v\right|^2$$
$$\leqslant \int_{-1}^{1}\left|\tfrac{1}{2}(j+1)u''_{m,n}(x,t)\right|^2|B_1|^2\,\mathrm{d}v$$
$$< \frac{M^2(j+1)^2}{(2i-3)(2i-1)^2},$$

where

$$B_1 = \left[\frac{P_{i+2}(v)-P_i(v)}{2i+3}-\frac{P_i(v)-P_{i-2}(v)}{2i-1}\right].$$

Thus we obtain

$$\left|\frac{1}{2}\int_{-1}^{1}u''_{m,n}(v)B_1\,\mathrm{d}v\right| < \frac{M(j+1)}{(2i-3)^{1/2}(2i-1)}$$
$$|c_{ij}| < \left|\frac{M(j+1)}{(2i-3)^{1/2}(2i-1)}\int_0^1 x^{\alpha-1}x^{j\alpha}\,\mathrm{d}x\right|$$
$$|c_{ij}| < \frac{M}{\alpha(2i-3)^{1/2}(2i-1)}.$$

Hence $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}$ is absolutely convergent and thus the expansion of the function given in (3) converges uniformly[19]. □

Theorem 2 *Let $u(x,t)$ be a function defined on $(0,1]$ with second derivative bounded by M (a constant). Then we have the following accuracy estimation:*

$$\epsilon \leqslant \sum_{j=n}^{\infty}\sum_{i=m}^{\infty}\frac{M}{\alpha^2(2i-1)\sqrt{(2i-3)(2i+1)(2j+1)}},$$

where $\epsilon = (\int_0^1\int_0^1[u(x,t)-u_{m,n}(x,t)]^2 dt\,dx)^{1/2}$.

Proof:

$$\epsilon^2 = \int_0^1\int_0^1[u(x,t)-u_{m,n}(x,t)]^2\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^1\sum_{j=n}^{\infty}\sum_{i=m}^{\infty}c_{ij}^2\mathscr{L}_i^{\alpha}(t)^2\phi_j^2(x^{\alpha})(xt)^{\alpha-1}\,\mathrm{d}t\,\mathrm{d}x.$$

Hence we have

$$\epsilon \leqslant \sum_{j=n}^{\infty}\sum_{i=m}^{\infty}\frac{M}{\alpha^2(2i-1)\sqrt{(2i-3)(2i+1)(2j+1)}}.$$

□

NUMERICAL RESULTS AND DISCUSSION

In this section, we present some examples to demonstrate the types of behaviour of the proposed method and the solution of a fractional partial differential equation.

Example 1 Consider the following non-homogeneous space-time fractional telegraph equation[20]

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} = \frac{\partial^2 u}{\partial t^2}+\frac{\partial^{\beta}u}{\partial t^{\beta}}+u-x^2-t+1,$$
$$x,\quad t \geqslant 0,\quad 1<\alpha\leqslant 2,\ 0<\beta\leqslant 1,$$

subject to $u(0,t)=t,\ u_x(0,t)=0$.

Let $u(x,t) = \sum_{j=0}^{2}(\sum_{i=0}^{4}c_{ij}\mathscr{L}_i^{\beta}(t))\Phi_j(x^{\alpha})$ be the approximate solution of Example 1 with $m=5$ and $n=3$.

By applying our proposed method, we have obtained the values of c_{ij} for various values of α and β (Table 1).

The numerical solutions of Example 1 obtained by our proposed method are shown in Fig. 1. It can be seen that the solution obtained by the present method is in full agreement with the exact solution $u(x,t)=x^2+t$ when $\alpha=2$ and $\beta=1$.

Example 2 Consider

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^{\beta}u}{\partial x^{\beta}},\quad 0<\alpha\leqslant 1,\ 1<\beta\leqslant 2,\ t>0,$$

subject to the initial condition $u(x,0)=x^2-x$ and the boundary conditions $u(0,t)=u(1,t)=2t$ for $t\geqslant 0$.

We apply our proposed method for solving Example 2 with the approximation $u(x,t) = \sum_{i=0}^{2}(\sum_{j=0}^{4}c_{ij}\mathscr{L}_i^{\alpha}(t))\Phi_j(x^{\beta})$. The numerical results are shown in Fig. 2 for various values of α and β. The solution obtained by our approach when $\alpha=1$ and $\beta=2$ converges to the exact solution $2t+x^2-x$.

Table 1 Values of the coefficients c_{ij} for Example 1.

$\alpha=2$ and $\beta=1$				$\alpha=1.5$ and $\beta=0.5$				$\alpha=1.25$ and $\beta=1$			
$i \searrow j$	0	1	2	$i \searrow j$	0	1	2	$i \searrow j$	0	1	2
0	1/2	1	0	0	1/3	$1/\Gamma(2.5)$	0	0	1	$1/\Gamma(2.25)$	0
1	1/2	0	0	1	1/2	0	0	1	0	0	0
2	0	0	0	2	1/6	0	0	2	0	0	0
3	0	0	0	3	0	0	0	3	0	0	0
4	0	0	0	4	0	0	0	4	0	0	0

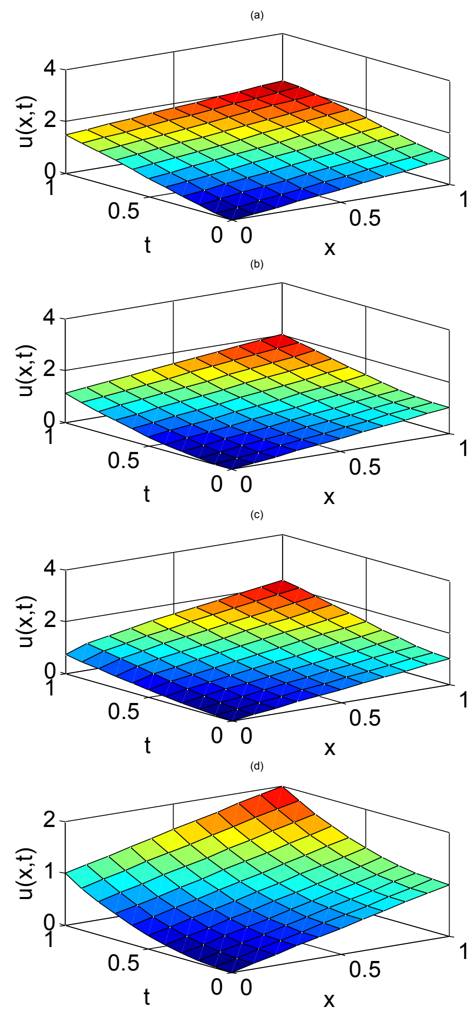

Fig. 1 Solution $u(x,t)$ to Example 1 when (a) $\alpha=1.25$, $\beta=1$; (b) $\alpha=1.75$, $\beta=1$; (c) $\alpha=1.5$, $\beta=0.5$; (d) $\alpha=2$, $\beta=1$.

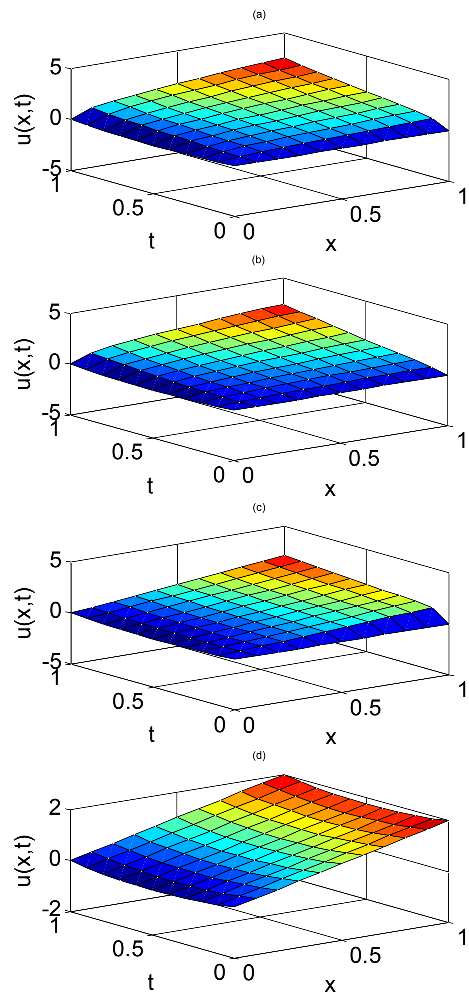

Fig. 2 Solution $u(x,t)$ to Example 2 when (a) $\alpha=0.5$, $\beta=1.5$; (b) $\alpha=0.5$, $\beta=1.25$; (c) $\alpha=1$, $\beta=1.75$; (d) $\alpha=1$, $\beta=2$.

CONCLUSION

The main concern of this work is to obtain an approximate solution of space-time fractional differential equations by a fractional shifted Legendre polynomial method. The applicability and validity of this method have been illustrated through theoretical analysis. From the illustrated examples, we observed that the obtained solutions are in good agreement with the solutions provided in the literature.

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