Generalization of the non-commuting graph of a group via a normal subgroup

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ABSTRACT: Let \( G \) be a finite group and \( N \) be a normal subgroup of \( G \). We define an undirected simple graph \( \Gamma_{N,G} \) to be a graph whose vertex set is all elements in \( G \setminus Z^N(G) \) and two vertices \( x \) and \( y \) are adjacent iff \( [x, y] \notin N \), where \( Z^N(G) = \{g \in G : [x, g] \in N \text{ for all } x \in G\} \). If \( N = 1 \), then we obtain the known non-commuting graph of \( G \). We give some basic results about connectivity, regularity, planarity, 1-planarity and some numerical invariants of the graph which are mostly improvements of the results given for non-commuting graphs. Also, a probability related to this graph is defined and a formula for the number of edges of the graph in terms of this probability is given.

KEYWORDS: commutativity degree, graph diameter, girth, strongly regular, planar graph

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INTRODUCTION

There are many graphs which are associated with groups: for instance, prime graphs\textsuperscript{1}, non-cyclic graphs\textsuperscript{2}, and conjugate graphs\textsuperscript{3}. An important one is the non-commuting graph. Let \( G \) be a group. Then the non-commuting graph of \( G \), denoted by \( \Gamma_G \), is a graph whose vertices are elements of \( G \setminus Z(G) \) and two distinct vertices \( x \) and \( y \) are adjacent if \( xy \neq yx \). Neumann\textsuperscript{4} obtained the first remarkable result on the non-commuting graphs by answering a problem of Erdős. From that time on, non-commuting graphs have been studied extensively in the literature\textsuperscript{5,6}.

The aim of this paper is to give a generalization of the non-commuting graph of a group \( G \) with respect to a given normal subgroup \( N \) of \( G \). In the next section, we introduce the generalized non-commuting graph \( G \) through a normal subgroup \( N \) of \( G \) denoted by \( \Gamma_{N,G} \) and state some of the basic graph theoretical properties of this graph which are mostly new or a generalization of some results in Ref. 5. We also give a connection between \( \Gamma_{N,G} \) and the probability that the commutator of two arbitrary elements of \( G \) belongs to the normal subgroup \( N \) of \( G \). We also state some conditions under which the graph is regular or strongly regular.

GENERALIZED NON-COMMUTING GRAPH

Definition 1 Let \( G \) be a finite group and \( N \) be a normal subgroup of \( G \). The non-commuting graph of \( G \) can be generalized using the subgroup \( N \) in such a way that its vertices are \( G \setminus Z^N(G) \) where \( Z^N(G) = \{g \in G : gn \in Z(G/N) \} \) and two distinct vertices are adjacent when \( [x, y] \notin N \). This graph is denoted by \( \Gamma_{N,G} \).

Notice that if \( N = 1 \) then the graph \( \Gamma_{N,G} \) is the ordinary non-commuting graph of \( G \). Clearly, \( N \subseteq Z^N(G) \) and so the elements of \( N \) do not belong to the vertex set of \( \Gamma_{N,G} \). One can easily see that the graph \( \Gamma_{N,G} \) is null whenever \( G \) is abelian or \( N = G \). Hence, throughout this paper, it is always assumed that \( G \) is a non-abelian group and \( N \) is a proper normal subgroup of \( G \).

We also note that if \( [G : N] = k \) and \( \{x_1, \ldots, x_k\} \) is a left transversal of \( N \) in \( G \) such that \( [x_i, x_j] \in N \) for all \( 1 \leq i, j \leq k \), then \( \Gamma_{N,G} \) is a null graph.

Example 1 Let \( D_{12} = \langle a, b \mid a^6 = b^2 = 1; bab = a^{-1} \rangle \) be the dihedral group of order 12 and put \( N = Z(D_{12}) = \{e, a^3\} \). Then \( \Gamma_{D_{12},N} \) is drawn as in Fig. 1.

Lemma 1 Let \( G \) be a group and \( N \) be a normal
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G is a non-abelian group and N is a proper normal subgroup of G. We also note that if \( n \in \mathbb{N} \), the result follows. Proposition 3 If \( |G| \) is odd, then \( N;G \) is Eulerian.

Let \( G \) be a non-abelian group and \( N \) be a proper normal subgroup of \( G \). Then \( \Gamma_{N,G} \) does not have any isolated vertex and is never an empty graph.

Definition 2 Let \( G \) be a group and \( N \) be a normal subgroup of \( G \). Then for every \( x \in G \),

\[
C_G^N(x) = \left\{ g \in G : gN \in C_{G/N}(xN) \right\}.
\]

It is clear that \( C_G^N(x) \) is a subgroup of \( G \) and \( \deg(x) = |G| - |C_G^N(x)| \) for all \( x \in V(\Gamma_{N,G}) \).

Theorem 1 Let \( G \) be a finite group and \( N \) be a nontrivial subgroup of \( G \). Then \( \Gamma_{N,G} \) is not a complete graph.

Proof: Suppose on the contrary that \( \Gamma_{N,G} \) is complete. Then

\[
\deg(h) = |G| - |C_G^N(h)| = |G| - |Z_G(N)| - 1
\]

for all vertices \( h \) of \( \Gamma_{N,G} \). Then \( |C_G^N(h)| = |Z_G(N)| + 1 \). On the other hand, \( |Z_G(N)| \) divides \( |C_G^N(h)| \). Thus \( |Z_G(N)| = 1 \) which contradicts the choice of \( N \).

Theorem 2 \( \text{diam}(\Gamma_{N,G}) = 2 \) and \( \text{girth}(\Gamma_{N,G}) = 3 \).

Proof: Let \( a, b \) be two non-adjacent vertices. Then there exist \( x, y \) such that \( [a, x] \notin N \) and \( [b, y] \notin N \). Now if \( a \) is adjacent to \( y \) or \( b \) is adjacent to \( x \), then \( d(x, y) = 2 \). Suppose \( [a, y] \in N \) and \( [b, x] \in N \). It is obvious that \( x \) is a vertex and therefore \( xy \) is adjacent to both \( x \) and \( y \). Hence \( \text{diam}(\Gamma_{N,G}) = 2 \).

Theorem 3 A lower bound for the minimum degree of vertices of \( \Gamma_{N,G} \) which we denote by \( \delta(\Gamma_{N,G}) \), satisfies \( \delta(\Gamma_{N,G}) \geq 3|N| \).

Proof: Let \( x \) be an arbitrary vertex of \( \Gamma_{N,G} \). Since \( \Gamma_{N,G} \) does not have any isolated vertex, there exists a vertex \( y \) adjacent to \( x \). Hence \( x \sim ny \) for all \( n \in N \). Since \( xy \) is a vertex of \( \Gamma_{N,G} \) adjacent to \( x \), we have \( x \sim nx \) for all \( n \in N \). As \( xy \neq yx \), \( x \sim nyx \) for all \( n \in N \), from which it follows that \( \delta(\Gamma_{N,G}) \geq 3|N| \).

Proposition 1 Let \( G \) be a non-abelian group and \( N \) be a normal subgroup of \( G \). Then \( \Gamma_{N,G} \) is Hamiltonian.

Proof: One can easily see that \( \deg(x) \geq |V(\Gamma_{N,G})|/2 \) for all \( x \in V(\Gamma_{N,G}) \). The proof is similar to that for Proposition 2.2 in Ref. 5.

An Eulerian tour is a walk which passes every edge exactly once. A graph is Eulerian if it contains an Eulerian tour. In the following propositions, we determine some conditions for \( N \) and \( G \) when the graph \( \Gamma_{N,G} \) is Eulerian.

Proposition 2 If \( |N| \) is even, then \( \Gamma_{N,G} \) is Eulerian.

Proof: Let \( V(\Gamma_{N,G}) = x_1N \cup \cdots \cup x_tN \). Let \( x \) be any vertex of \( \Gamma_{N,G} \) and \( x_i, \ldots, x_t \) be those vertices among \( x_1, \ldots, x_t \) adjacent to \( x \). Hence \( x \sim nx_i \) for all \( n \in N \) and \( 1 \leq i \leq t \), which implies that \( \deg(x) = t|N| \) is even. Hence by Theorem 4.1 in Ref. 7 the result follows.

Proposition 3 If \( |G| \) is odd, then \( \Gamma_{N,G} \) is Eulerian.

Proof: Let \( x \in V(\Gamma_{N,G}) \) be an arbitrary vertex. We know that \( \deg(x) = |G| - |C_G^N(x)| \). Since \( C_G^N(x) \) is a subgroup of \( G \), its order is odd and hence \( \deg(x) \) is even. The result follows by Theorem 4.1 in Ref. 8.

In the following theorem, we give a lower bound for the chromatic number \( \chi(\Gamma_{N,G}) \) of the graph \( \Gamma_{N,G} \).

Theorem 4 Suppose \( [G : N] = k+1 \) and \( G = x_0N \cup x_1N \cup \cdots \cup x_kN \). Then

\[
\chi(\Gamma_{N,G}) \geq 1 + \max|A| : A \subseteq \{x_0, \ldots, x_k\},
\]

\[
x_i, x_j \in A, [x_i, x_j] \notin N, \}
\]

Proof: Put \( t_1 = |\{x_i : [x_i, x_1] \notin N\}| \). Then it is clear that \( \chi(\Gamma_{N,G}) \geq t_1 + 1 \). Similarly, if \( t_j = |\{x_i : [x_i, x_j] \notin N\}| \), then we need at least \( t_1 + t_j + 1 \) colours in order to colour \( \Gamma_{N,G} \) and again \( \chi(\Gamma_{N,G}) \geq t_1 + 1 \). Now, if \( t = \max\{t_1 : 1 \leq i \leq k\} \), then \( \chi(\Gamma_{N,G}) \geq t + 1 \), as required.

We now may state some results for the dominating number of the graph \( \Gamma_{N,G} \). A subset of the graph is a dominating set if every vertex which is not in the subset is adjacent to at least one member of the subset. The size of the minimum dominating set is called the dominating number.
Theorem 5 Let $G$ be a non-abelian group and $N$ be a normal subgroup of $G$ such that $xN$ is a dominating set for $\Gamma_{N,G}$ for every vertex $x$ of $\Gamma_{N,G}$. Then $Z^N(G) = \{x \mid x^2 = 1, \text{ and } c_{G/N}(x) = \langle xN \rangle \}$.

Proof: To prove $Z^N(G) = \{x \mid x^2 = 1, \text{ and } c_{G/N}(x) = \langle xN \rangle \}$, assume on the contrary that $Z^N(G) = \{x \mid x^2 = 1\}$. Then there exists an element $a \in Z^N(G)$ such that $a \notin N$ and so $\{a, g\} \notin N$ for all $g \in G$. Since $x$ is a vertex, $x \notin Z^N(G)$ which implies that $[x, b] \notin N$ for some $b \in G$. Thus $[xa, x] \notin N$. Consequently, $xa$ is a vertex. We now have $\{xa, x\} \notin N$ and so $xa$ is not adjacent to $x$ which is a contradiction. If $x^2 = 1$, then $x^{-1} = x$ and again $x^{-1}$ is not adjacent to $x$ is a contradiction. The last part directly follows from the point that a path between $x_1$ and $y_1$ which is a contradiction. Hence $t > 1$. \hfill $\square$

The commutativity degree $d(G)$ of a finite group $G$ is the probability that two randomly chosen elements of $G$ commute. We may extend it to the commutativity degree of $G$ with respect to a normal subgroup $N$ of $G$, denoted by $d^N(G)$:

$$d^N(G) = \frac{|\{(x, y) \in G \times G : [x, y] \in N\}|}{|G|^2}.$$ 

It is obvious that if $N = 1$, then $d(G) = d^N(G)$. Also, one can see that $d^N(G) = d(G/N)$. Utilizing the above definition, we may find a formula for the number of edges of $\Gamma_{N,G}$.

Lemma 2 The number of edges of $\Gamma_{N,G}$ is $\frac{1}{2}|G|^2(1 - d^N(G))$.

Proof: Let $A = \{(x, y) \in G \times G \mid [x, y] \in N\}$ and $B = \{(x, y) \in G \times G \mid [x, y] \notin N\}$. It is easy to see that $d^N(G) = |A|/|G|^2$. On the other hand,

$$|G|^2 = |\{(x, y) \in G \times G : [x, y] \in N\}| + |\{(x, y) \in G \times G : [x, y] \notin N\}| = |A| + |B|.$$ 

Also, as $|B| = 2|E(\Gamma_{N,G})|$, we have

$$2|E(\Gamma_{N,G})| = |B| = |G|^2 - |A| = |G|^2(1 - d^N(G))$$

and the result follows. \hfill $\square$

Using the above lemma, we obtain the following inequalities.

Proposition 5 Let $G$ be a finite group and $N$ be a normal subgroup of $G$. Then

(i) $d^N(G) \geq \frac{1}{2}|Z^N(G)|/|G| + 1/|G| - |Z^N(G)|/|G|^2$;

(ii) $|E(\Gamma_{N,G})| \geq \frac{3}{4}|G|^2$, if $G/N$ is not abelian.

Proof: They follow from Lemma 2 and an upper bound $\frac{5}{8}$ for $d(G)$ by a result in Ref. 9. \hfill $\square$

A graph is $k$-regular if every vertex has degree $k$. Also, a $k$-regular graph with $n$ vertices is called strongly regular with parameter $(n, k, r, s)$ if it is neither empty nor complete such that any two adjacent vertices are adjacent to $r$ common vertices and any two non-adjacent vertices are adjacent to $s$ common vertices.

Theorem 6 If $\Gamma_{N,G}$ is a $k$-regular graph, then $|N|$ divides $k$.

Proof: Let $x \in V(\Gamma_{N,G})$ and $y$ be a vertex adjacent to $x$. Then $x \sim y\Gamma$ for all $y \in N$ so that $|N|$ divides $\text{deg}(x)$. Since $\text{deg}(x) = k$, it follows that $|N|$ divides $k$, as required. \hfill $\square$
Theorem 7 If $\Gamma_{N,G}$ is strongly regular with parameters $(n, k, r, s, t)$, then $s = k$.

Proof: Let $x \in V(\Gamma_{N,G})$ and $z \in Z^N\backslash N$. Clearly, $xz \in V(\Gamma_{N,G})$. Since $x$ and $xz$ are not adjacent, $s = |G \backslash (C^N_N(x) \cup C^N_N(xz))]$. Since $C^N_N(x) = C^N_N(xz)$ and $\Gamma_{N,G}$ is $k$-regular, the result follows. □

To prove the next theorem, we need the following lemma concerning AC-groups. A group is an AC-group if its nontrivial centralizers are abelian.

Lemma 3 The following statements are equivalent:

(i) $G$ is an AC-group;
(ii) if $x, y \in G \backslash Z(G)$ such that $[x, y] = 1$, then $C_G(x) = C_G(y)$;
(iii) if $x, y, z \in G \backslash Z(G)$ such that $[x, y] = [y, z] = 1$, then $[x, z] = 1$;
(iv) if $A$ and $B$ are subgroups of $G$ such that $Z(G) \subseteq C_G(A) \subseteq C_G(B) \subset G$, then $C_G(A) = C_G(B)$.

Proof: See Lemma 3.2 in Ref. 10. □

Corollary 1 If $\Gamma_{N,G}$ is a strongly regular graph with parameters $(n, k, r, s, t)$, then $G/N$ is an AC-group.

Proof: Let $x, y$ be two non-adjacent vertices. Then $C_{G/N}(xN) = C_{G/N}(yN)$ since $\deg(x) = |G \backslash C^N_G(x)|$ and $\deg(y) = |G \backslash C^N_G(y)|$ and hence $|C^N_G(x)| = |C^N_G(y)|$. As $k = |G - C^N_G(x)| = |N(x) \cap N(y)| = |G \backslash C^N_G(x) \cup C^N_G(y)|$, we conclude that $|C^N_G(x)| = |C^N_G(x) \cup C^N_G(y)|$. Hence $C^N_G(x) \subseteq C^N_G(x)$ and consequently $C^N_G(x) = C^N_G(y)$. Thus $C^N_G(x)/N = C^N_G(y)/N$. Hence $C_{G/N}(N) = C_{G/N}(yN)$ and, by Lemma 3, $G/N$ is an AC-group. Let $x, y$ be two non-adjacent vertices. Then $[x, y] \in N$. Since $\deg(x) = |G \backslash C^N_G(x)| = |G \backslash C^N_G(y)|$ and $\Gamma_{N,G}$ is regular, we have $|C^N_G(x)| = |C^N_G(y)|$. Assume that $t \in G \backslash (C^N_G(x) \cup C^N_G(y))$. Then $t$ is adjacent to $x$ and $y$. The number of common neighbours of $x$ and $y$ are equal and is $k$. Therefore, we should have $k = |G \backslash (C^N_G(x) \cup C^N_G(y))| = |G \backslash C^N_G(x)|$. Consequently, $|C^N_G(x)| = |C^N_G(x) \cup C^N_G(y)|$ and so $C^N_G(x) = C^N_G(y)$. Hence $C^N_G(xN) = C^N_G(yN)$ and by Lemma 3, $G/N$ is an AC-group as required. □

Theorem 8 If $\Gamma_{N,G}$ is a strongly regular graph with parameters $(n, k, r, s)$ then $r = 2k - n$.

Proof: Let $x, y$ be two adjacent vertices. Then $[x, y] \notin N$. Hence, $C^N_G(x) \cap C^N_G(y) = Z^N(G)$, by Corollary 1 and a consequence of Lemmas 4.2 and 4.3 in Ref. 5. We now have

$$r = |N(x) \cap N(y)|$$

$$= |G \backslash C^N_G(x) \cup C^N_G(y)|$$

$$= |G| - |C^N_G(x)| - |C^N_G(y)| + |C^N_G(x) \cap C^N_G(y)|$$

$$= |G| - |C^N_G(x)| + |G| - |C^N_G(y)| + |Z^N(G) - |G||$$

$$= 2k - n,$$

where $N(x)$ is the set of neighbours of $x$. □

In the following two propositions, we give a link between this graph and the nilpotency property in group theory.

Proposition 6 Suppose $G = x_1N \cup x_2N \cup \ldots \cup x_kN$ such that $x_i$ is a vertex in $\Gamma_{N,G}$, for each $2 \leq i \leq k$. Then $G$ is not nilpotent.

Proof: Since all $x_i$ for $2 \leq i \leq k$ are vertices, $Z^N(G) = N$ and therefore $|Z(G/N)| = 1$. Hence $G$ cannot be nilpotent. □

Proposition 7 Let $N$ be a normal subgroup of $G$ and $G/N$ be non-nilpotent. If $\Gamma_{N,G}$ is regular, then $G/N$ is a nilpotent group.

Proof: For every vertex $x$ in $\Gamma_{N,G}$ we have $\deg(x) = |G - C^N_G(x)|$. Moreover, if $y \notin Z^N(G)$, then $\deg(x) = \deg(y)$ which implies that $|C^N_G(x)| = |C^N_G(y)|$. On the other hand, if $zN \in Z(G/N)$, then $C_{G/N}(zN) = G/N$ and consequently, the size of all conjugacy classes of $G/N$ has only two values: 1 and a positive integer $n \neq 1$. Hence $G/N$ is a nilpotent by a theorem of Ito. □

Recall that a graph is planar if it can be drawn in the plane such that edges intersect only at vertices. The planarity of $\Gamma_{N,G}$ is investigated in Ref. 5 when $N = 1$ is the trivial subgroup. In the following theorem, we show that $\Gamma_{N,G}$ is not planar when $N \neq 1$.

Theorem 9 The graph $\Gamma_{N,G}$ is never planar for every nontrivial normal subgroup $N$ of $G$.

Proof: Let $N \neq 1$ and $\Gamma_{N,G}$ be planar. Then $\delta(\Gamma_{N,G}) \leq 5$ (see Corollary 3.5.9 in Ref. 8). On the other hand, $\delta(\Gamma_{N,G}) \geq 3|N|$ by Theorem 3. Hence $3|N| \leq 5$, which is a contradiction. □

A graph is called 1-planar if it can be drawn in the plane in such a way that each edge is crossed by no more than one other edge. Note that graphs $K_7$, $K_{3,7}$, and $K_{4,5}$ are not 1-planar in Ref. 12.

Lemma 4 There is a vertex in $\Gamma_{N,G}$ whose order in $G$ is greater than 2.
Fig. 2 Induced subgraph of \( \Gamma_{N,G} \) isomorphic to \( K_{3,7} \).

Fig. 3 Induced subgraphs of \( \Gamma_{N,G} \) isomorphic to \( K_{4,5} \).

Hence it is not a 1-planar graph. Also, if \( G \cong T \), then \( \Gamma_{N,G} \) has a subgraph isomorphic to \( K_{4,5} \) as drawn in the right of Fig. 3. Hence it is not 1-planar.

Finally, if \( G \cong D_{14} \), then \( \Gamma_{N,G} \) has a subgraph isomorphic to \( K_{7} \) so that it is not a 1-planar graph. \( \square \)

A graph is called outer planar if it can be drawn in the plane without crossing edges in such a way that all vertices belong to the unbounded face of the drawing.

**Theorem 11** The graph \( \Gamma_{N,G} \) is never an outer-planar graph.

**Proof:** If \( |N| \geq 2 \), then clearly \( \Gamma_{N,G} \) is not planar by Theorem 9 and consequently is not outer planar. Thus we may assume that \( N = 1 \) and that \( \Gamma_{N,G} \) is outer planar. Hence \( w(\Gamma_{N,G}) \leq 3 \). As in the proof of Proposition 2.3 in Ref. 5, one can show that \( |G| \leq 8 \). The only non-abelian groups of order less or equal to 8 are \( S_{3}, D_{8} \) and \( Q_{8} \) for which the graph \( \Gamma_{N,G} \) has a subgraph isomorphic to \( K_{2,3} \), which is a contradiction. Hence \( \Gamma_{N,G} \) is not an outer planar graph. \( \square \)

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