Superconvergence of triangular mixed finite element methods for nonlinear optimal control problems

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ABSTRACT: In this paper, we investigate the superconvergence of nonlinear elliptic optimal control problems by using triangular mixed finite element methods. The state and the co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We obtain the superconvergence of $O(h^{3/2})$ for the control variable and coupled state variable. Numerical results demonstrating these superconvergence results are also presented.

KEYWORDS: lowest order Raviart-Thomas mixed finite element methods, triangular partition

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INTRODUCTION

Optimal control problems play increasingly important role in multi-disciplinary applications such as engineering design, fluid mechanics, physical, biological, medicine, finance, and social-economic systems. There are various numerical methods to solve these complex problems. Among these numerical methods, finite element methods for state equations have many applications. Papers devoted to linear-quadratic optimal control problems include those by Falk\textsuperscript{1} and Geveci\textsuperscript{2}. The authors studied the numerical approximation of distributed nonlinear optimal control problems with pointwise constraints on the control\textsuperscript{3}. Meyer and Rösch\textsuperscript{4} analysed finite element discretization of the dimensional (2-d) elliptic optimal control problem. These approximations have convergence of order $h^2$. A posteriori error estimates for distributed convex optimal control problems and nonlinear optimal control problems have been obtained\textsuperscript{5,6}.

Recently, we obtained a priori error estimates and a posteriori error estimates of mixed finite element methods for linear and nonlinear optimal control problems\textsuperscript{7–9}. Then we used the post-processing projection operator to prove a quadratic superconvergence of the control for linear elliptic optimal control problem by a mixed finite element method\textsuperscript{10–12}.

We are concerned with the 2-d nonlinear elliptic optimal control problem

\begin{equation}
\min_{u \in U_{ad}} \left\{ \frac{1}{2} \| \mathbf{p} - \mathbf{p}_d \|^2 + \frac{1}{2} \| y - y_d \|^2 + \frac{\alpha}{2} \| u \|^2 \right\}
\end{equation}

subject to the state equations

\begin{equation}
\text{div} \mathbf{p} + \phi(y) = u, \quad \mathbf{p} = -A \nabla y, \quad x \in \Omega,
\end{equation}

with the boundary condition

\begin{equation}
y = 0, \quad x \in \partial \Omega,
\end{equation}

where $\Omega$ is a rectangular domain, $\mathbf{p}_d$ and $y_d$ are two known functions, $\mathbf{p}$ and $y$ are state variables, $u$ is a...
We shall construct a discretized scheme for the (1)–(3) is to find $\nu > 0$ is a constant. We denote the set of admissible controls by $U_{ad}$, where

$$U_{ad} = \{ u \in L^2(\Omega) : u \geq 0 \text{ a.e. in } \Omega \}.$$ 

Let us state the assumptions on the operator $A$ and the functional $\phi$: (A1) the coefficient matrix function $A(x) = (a_{ij}(x))$ is symmetric with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition $c_1\|\xi\|^2 \leq \sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \leq c_c\|\xi\|^2$, $\forall (\xi, x) \in \mathbb{R}^2 \times \Omega$; $c_c, c_c > 0$; (A2) $\phi$ is of class $C^2$ with respect to the variable $y$, for any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(R, R)$, $\phi(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq \lambda > 0$.

**Lemma 1 (Ref. 13)** For every function $g \in L^p(\Omega)$ ($p \geq 1$), the solution $y$ of

$$-\text{div}(A\nabla y) + \phi(y) = g \text{ in } \Omega, \quad y|_{\partial \Omega} = 0,$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant $C$ such that

$$\|y\|_{W^{2,p}(\Omega)} \leq C \|g\|_{L^p(\Omega)}.$$

Next, we introduce the co-state elliptic equations

$$\text{div} q + \phi'(y)z = y - y_d, \quad q = -A(\nabla z + p - p_d),$$

with boundary condition $z = 0$, $x \in \partial \Omega$. The existence of a unique solution of (2) and (4) is justified by Lemma 1. Furthermore, we make the following realistic assumption (A3): $u \in W^{1,\infty}(\Omega)$, $y, z \in H^2(\Omega)$.

**MIXED METHODS FOR OPTIMAL CONTROL PROBLEM**

We shall construct a discretized scheme for the nonlinear optimal control problem (1)–(3) by using mixed finite element methods and give its equivalent optimality conditions.

Let $W = L^2(\Omega)$,

$$V = H(\text{div}; \Omega) = \{ v \in L^2(\Omega)^2, \text{div } v \in L^2(\Omega) \}. $$

The Hilbert space $V$ is equipped with the following norm: $\|v\|_{\text{div}} = \|v\|_{H(\text{div}; \Omega)} = (\|v\|^2 + \|\text{div } v\|^2)^{1/2}$. A weak formulation of the optimal control problem (1)–(3) is to find $(p, y, u) \in V \times W \times U_{ad}$ such that

$$\min_{p \in U_{ad}} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\alpha}{2} \|u\|^2 \right\}$$

$$\min_{p \in U_{ad}} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\alpha}{2} \|u\|^2 \right\}$$

$$\min_{p \in U_{ad}} \left\{ \frac{1}{2} \|p - p_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\alpha}{2} \|u\|^2 \right\}$$

is to find $u \in U_{ad}$ such that

$$\begin{align*}
(A^{-1}p, v) - (y, \text{div } v) &= 0, \quad \forall v \in V, \\
\text{div } (p, w) + (\phi(y), w) &= (u, w), \quad \forall w \in W, \\
\end{align*}$$

where the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ is denoted by $(\cdot, \cdot)$. It is well known that the convex control problem (5)–(7) has a solution $(p^*, y^*, u^*)$, and that if a triplet $(p^*, y^*, u^*) \in V \times W \times U$ is the solution of (5)–(7), then there exists a co-state $(q^*, z^*) \in V \times W$ such that $(p^*, y^*, q^*, z^*, u^*)$ satisfies the following optimality conditions:

$$\begin{align*}
(A^{-1}p^*, v) - (y^*, \text{div } v) &= 0, \\
\text{div } (p^*, w) + (\phi(y^*), w) &= (u^*, w), \\
(A^{-1}q^*, v) - (z^*, \text{div } v) &= -(p^* - p_d, v), \\
\text{div } q^*, w) + (\phi'(y^*)z^*, w) &= (y^* - y_d, w), \\
(z^* + au^*, \bar{u} - u^*) &= 0,
\end{align*}$$

where $v \in V, w \in W$ and $\bar{u} \in U_{ad}$.

We now introduce the discretized problem by considering a family of triangulations $\mathcal{T}_h$ of $\Omega$. With each element $T_i \in \mathcal{T}_h$, we associate two parameters $\rho(T_i)$ and $\sigma(T_i)$, where $\rho(T_i)$ denotes the diameter of the set $T_i$ and $\sigma(T_i)$ is the diameter of the largest ball contained in $T_i$. The mesh size of the grid is defined by $h = \max_{T_i \in \mathcal{T}_h} \rho(T_i)$. We suppose that the following regularity assumptions are satisfied. There exist two positive constants $\varrho_1$ and $\varrho_2$ such that $(\rho(T_i)/\sigma(T_i)) \leq \varrho_1$, $(h/\rho(T_i)) \leq \varrho_2$ for all $T_i \in \mathcal{T}_h$ and all $h > 0$.

Let $V_h \times W_h \subset V \times W$ denote the lowest order Raviart-Thomas mixed finite element space, namely,

$$V_h = \{ v \in V : \forall T_i \in \mathcal{T}_h, \phi|_{T_i} \in P_0(T_i), \phi|_{T_i} = \phi|_{T_j} \text{ on } \partial T_i \},$$

$$W_h = \{ w \in W : \forall T_i \in \mathcal{T}_h, w|_{T_i} \in P_0(T_i), \}$$

where $P_0(T_i)$ indicates a constant on $T_i$. To approximate the control, we use the following cone of nonnegative piecewise constant functions:

$$U_h = \{ \bar{u} \in U_{ad} : \bar{u}|_{T_i} = \text{constant}, \quad \forall T_i \in \mathcal{T}_h \}.$$
where \( \|v\|_{r,\rho} \) denotes the norm of the usual Sobolev space \( W^{r,\rho}(\Omega) \) for \( 1 \leq \rho \leq +\infty \) and \( r \geq 0 \).

The mixed finite element approximation of (5)–(7) is to find \((p_h,y_h,u_h) \in V_h \times W_h \times U_h \) such that

\[
\min_{u \in U_h} \frac{1}{2} \left\{ \|p_h - p_0\|_V^2 + \|y_h - y_d\|_V^2 + \alpha \|u_h\|_H^2 \right\}
\]

(19)

\[
(A^{-1} p_h, v_h) - (y_h, \textrm{div } v_h) = 0,
\]

(20)

\[
(\text{div } p_h, w_h) + (\phi(y_h), w_h) = (u_h, w_h),
\]

(21)

where \( v_h \in V_h \) and \( w_h \in W_h \).

The optimal control problem (19)–(21) again has a solution \((p_h^*, y_h^*, u_h^*) \in V_h \times W_h \times U_h \) if a triplet \((p_h^*, y_h^*, u_h^*) \in V_h \times W_h \times U_h \) is a co-state \((q_h^*, z_h^*, \tau_h^*) \in V_h \times W_h \times U_h \) such that \((p_h^*, y_h^*, q_h^*, z_h^*, \tau_h^*) \) satisfies the following discretized optimality conditions:

\[
(A^{-1} p_h^*, v_h) - (y_h^*, \textrm{div } v_h) = 0,
\]

(22)

\[
(\text{div } p_h^*, w_h) + (\phi(y_h^*), w_h) = (u_h^*, w_h),
\]

(23)

\[
(A^{-1} q_h^*, v_h) - (z_h^*, \textrm{div } v_h) = -(p_h^* - p_0, v_h),
\]

(24)

\[
(\text{div } q_h^*, w_h) + (\phi(y_h^*), w_h) = (y_h^* - y_d, w_h),
\]

(25)

\[
(z_h^* + \alpha u_h^*, \tilde{u}_h - u_h^*) \geq 0,
\]

(26)

where \( v_h \in V_h \), \( w_h \in W_h \) and \( \tilde{u}_h \in U_h \).

We shall use some intermediate variables. For any control function \( \bar{u} \in U_{ad} \), we define the state solution \((p^*(\bar{u}), y^*(\bar{u}), q^*(\bar{u}), z^*(\bar{u}))\) associated with \( \bar{u} \) which satisfies

\[
(A^{-1} p^*(\bar{u}), v) - (y^*(\bar{u}), \textrm{div } v) = 0,
\]

(27)

\[
(\text{div } p^*(\bar{u}), w) + (\phi(y^*(\bar{u})), w) = (\bar{u}, w),
\]

(28)

\[
(A^{-1} q^*(\bar{u}), v) - (z^*(\bar{u}), \textrm{div } v) = -(p^*(\bar{u}) - p_0, v),
\]

(29)

\[
(\text{div } q^*(\bar{u}), w) + (\phi(y^*(\bar{u})), w) = (y^*(\bar{u}) - y_d, w),
\]

(30)

where \( v \in V \) and \( w \in W \). We define the discrete state solution \((p_h^*(\bar{u}), y_h^*(\bar{u}), q_h^*(\bar{u}), z_h^*(\bar{u}))\) corresponding to \( \bar{u} \) which satisfies

\[
(A^{-1} p_h^*(\bar{u}), v_h) - (y_h^*(\bar{u}), \textrm{div } v_h) = 0,
\]

(31)

\[
(\text{div } p_h^*(\bar{u}), w_h) + (\phi(y_h^*(\bar{u})), w_h) = (\bar{u}, w_h),
\]

(32)

\[
(A^{-1} q_h^*(\bar{u}), v_h) - (z_h^*(\bar{u}), \textrm{div } v_h) = -(p_h^*(\bar{u}) - p_0, v_h),
\]

(33)

\[
(\text{div } q_h^*(\bar{u}), w_h) + (\phi(y_h^*(\bar{u})), z_h^*(\bar{u}), w_h) = (y_h^*(\bar{u}) - y_d, w_h),
\]

(34)

where \( v_h \in V_h \) and \( w_h \in W_h \). With these definitions, the exact state solution and its approximation can be written as

\[
(p^*, y^*, q^*, z^*) = (p^*(u^*), y^*(u^*), q^*(u^*), z^*(u^*)),
\]

\[
(p_h^*, y_h^*, q_h^*, z_h^*) = (p_h^*(u_h^*), y_h^*(u_h^*), q_h^*(u_h^*), z_h^*(u_h^*)).
\]

For \( \varphi \in W_h \), we shall write

\[
\hat{\phi}(\varphi) - \phi(\varphi) = -\hat{\phi}'(\varphi)(\rho - \varphi) = -\hat{\phi}'(\varphi)(\rho - \varphi) + \hat{\phi}''(\varphi)(\rho - \varphi)^2,
\]

(35)

where \( \hat{\phi}'(\varphi) = \int_0^1 \hat{\phi}'(\varphi + t(\rho - \varphi))dt, \hat{\phi}''(\varphi) = \int_0^1 (1-t)\hat{\phi}''(\rho + t(\rho - \varphi))dt \) are bounded functions in \( \Omega \).

SUPERCONVERGENCE

Firstly, we can obtain the following technical results.

**Lemma 2** Suppose (A1) hold. Let \( \gamma \in C^1(\Omega), \omega \in V, \varphi \in L^2(\Omega)^2, \) and \( \psi \in L^2(\Omega) \). If \( \tau \in W_h \) satisfies

\[
(A^{-1} \omega, v_h) - (\tau, \textrm{div } v_h) = (\varphi, v_h), \quad \forall \, v_h \in V_h,
\]

\[
(\text{div } \omega, w_h) + (\gamma \tau, w_h) = (\psi, w_h), \quad \forall \, w_h \in W_h,
\]

then there exists a constant \( C \) such that

\[
\|\tau\|_0 \leq C\left( h\|\omega\|_0 + h^2 \|\text{div } \omega\|_0 + \|\varphi\|_0 + \|\psi\|_0 \right),
\]

(36)

for \( h \) sufficiently small.

For any \( \bar{u} \in U \), let \( \epsilon_1 := p^*(\bar{u}) - p_h^*(\bar{u}), \epsilon_1 := y^*(\bar{u}) - y_h^*(\bar{u}) \).

To analyse the intermediate errors, let us first note the following error equations from (22)–(23) and (27)–(28):

\[
(A^{-1} \epsilon_1, v_h) - (\epsilon_1, \textrm{div } v_h) = 0,
\]

(37)

\[
(\text{div } \epsilon_1, w_h) + (\hat{\phi}'(y^*(\bar{u}))\epsilon_1, w_h) = 0,
\]

(38)

where \( v_h \in V_h \) and \( w_h \in W_h \).

By using Lemma 2, we can establish the following error estimates:
Lemma 3 Suppose that assumptions (A1–3) are fulfilled. Let \( y^*(\tilde{u}) \) and \( y_h^*(\tilde{u}) \) be the solutions of (27)–(30) and (31)–(34), respectively. If the intermediate solutions satisfy \( p^*(\tilde{u}) \in [H^2(\Omega)]^2, \ q^*(\tilde{u}) \in H^4(\Omega) \), then we have
\[
\|y_h^*(\tilde{u}) - y^*(\tilde{u})\| + \|p_h^*(\tilde{u}) - p^*(\tilde{u})\|_{\text{div}} \leq Ch^2. \tag{39}
\]

Lemma 4 Suppose that assumptions (A1–3) are fulfilled. Let \( z^*(\tilde{u}) \) and \( z_h^*(\tilde{u}) \) be the solutions of (27)–(30) and (31)–(34), respectively. If the intermediate solutions satisfy \( p^*(\tilde{u}), q^*(\tilde{u}) \in [H^2(\Omega)]^2, \ y^*(\tilde{u}), z^*(\tilde{u}) \in H^4(\Omega) \), then
\[
\|z_h^*(\tilde{u}) - z^*(\tilde{u})\| + \|q_h^*(\tilde{u}) - q^*(\tilde{u})\|_{\text{div}} \leq Ch^2. \tag{40}
\]

Lemma 5 Suppose that assumptions (A1–3) are valid. Let \( P_h u \) be the local \( L^2(\Omega) \) projection of the exact control \( u \) and \( z^*(P_h u) \) and \( z^*(u) \) be the solutions of (27)–(30) with \( \tilde{u} = P_h u \) and \( \tilde{u} = u \), respectively. Then we have
\[
\|z^*(P_h u) - z^*(u)\| \leq Ch^2. \tag{41}
\]

Let \((p^*(u^*), y^*(u^*))\) and \((p_h^*(u_h^*), y_h^*(u_h^*))\) be the solutions of (8)–(12) and (22)–(26), respectively. Let \( J(\cdot) : U \rightarrow \mathbb{R} \) be a \( G \)-differential convex functional with the following form:
\[
J(u^*) = \frac{1}{2} \|p^* - p_0\|^2 + \frac{1}{2} \|y^* - y_0\|^2 + \frac{\alpha}{2} \|u^*\|^2.
\]

It can be shown that
\[
(J'(u^*), v) = (z^* + au^*, v), \quad (J'(u_h^*), v) = (z_h^*(u_h^*) + au_h^*, v).
\]

In many applications, \( J(\cdot) \) is uniform convex near the solution \( u^* \) (see Ref. 5). Then there is a \( c > 0 \), independent of \( h \), such that
\[
(J'(u^*) - J'(u_h^*)) \geq c \|u^* - u_h^*\|^2, \tag{42}
\]
where \( u^* \) and \( u_h^* \) are the solutions of (12) and (26), respectively. The convexity of \( J(\cdot) \) is closely related to the second order sufficient conditions of the optimal control problem, which are assumed in many studies on numerical methods of the problem.

Let
\[
\Omega^+ = \left\{ |T_i : T_i \in \Omega, u^*|_{T_i} > 0 \right\}, \quad \Omega^0 = \left\{ |T_i : T_i \in \Omega, u^*|_{T_i} = 0 \right\},
\]
\[
\Omega^b = \Omega \setminus (\Omega^+ \cup \Omega^0).
\]

We will assume that \( u^* \) and \( \mathcal{G}_h \) are regular such that \( |\Omega^b| \leq Ch \). We are now able to obtain our first main result.

**Theorem 1** Suppose that assumptions (A1–3) are satisfied. Let \( P_h u \) be the local \( L^2(\Omega) \) projection of the exact control \( u \) and \( u_h^* \) be the solution of (22)–(26). Then we have the estimate
\[
\|P_h u^* - u_h^*\| \leq Ch^{3/2}. \tag{43}
\]

Proof: Let \( v = u_h^* \) in (12) and \( v_h = P_h u^* \) in (26). We have
\[
(z^* + au^*, u_h^* - u_h^*) \geq 0, \quad (z_h^* + au_h^*, P_h u^* - u_h^*) \geq 0.
\]

Adding the two inequalities gives
\[
(z_h^* + au_h^* - z^* - au^*, P_h u^* - u_h^*) + (z^* + au^*, P_h u^* - u_h^*) \geq 0.
\]

So we obtain
\[
\alpha(P_h u^* - u_h^*, P_h u^* - u_h^*) \geq (z_h^* - z^*(u_h^*), P_h u^* - u_h^*) + (z^* + au^*, P_h u^* - u_h^*). \tag{44}
\]

Clearly,
\[
(z_h^* - z^*, P_h u^* - u_h^*) = (z_h^* - z^*(u_h^*), P_h u^* - u_h^*) + (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*) + (z^* + au^*, P_h u^* - u_h^*). \tag{45}
\]

Then
\[
\alpha(P_h u^* - u_h^*, P_h u^* - u_h^*) - (z_h^*(u_h^*) - z^*(P_h u^*), P_h u^* - u_h^*) \leq (z_h^* - z^*(u_h^*), P_h u^* - u_h^*) + (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*) + (z^* + au^*, P_h u^* - u^*).
\]

Now we find bounds for the \( E_i \). From Lemma 4, we have
\[
E_1 = (z_h^* - z^*(u_h^*), P_h u^* - u_h^*) \leq C \|z_h^* - z^*(u_h^*)\| \cdot \|P_h u^* - u_h^*\| \leq Ch^2 \|P_h u^* - u_h^*\|. \tag{46}
\]

From (40),
\[
E_2 = (z^*(P_h u^*) - z^*(u^*), P_h u^* - u_h^*) \leq C \|z^*(P_h u^*) - z^*(u^*)\| \cdot \|P_h u^* - u_h^*\| \leq Ch^2 \|P_h u^* - u_h^*\|. \tag{47}
\]
Finally,\[
E_3 = (z^* + au^*, P_h u^* - u^*) \\
= \int_{\Omega^0} (z^* + au^*, P_h u^* - u^*) \, dx \\
+ \int_{\Omega^1} (z^* + au^*, P_h u^* - u^*) \, dx \\
= \int_{\Omega^0} (z^* + au^*, P_h u^* - u^*) \, dx. \quad (48)
\]

From the definition of \(\Omega^0\) we note that \((P_h u^* - u^*)|_{\Omega^0} = 0\). It is clear that
\[
\int_{\Omega^0} (z^* + au^*, P_h u^* - u^*) \, dx = 0. \quad (49)
\]

From (12), we have pointwise a.e. \(z^* + au^* \geq 0\). We choose \(\bar{u}|_{\Omega^1} = 0\) and \(\bar{u}|_{\Omega^0} = u^*\) so that \((z^* + au^*, u^*)|_{\Omega^1} \leq 0\). Hence \((z^* + au^*)|_{\Omega^1} = 0\). Let \(\pi^* u^*\) be the integral operator such that \(\pi^* u^*|_{\Omega^1} = \int_{\Omega^1} u^*/\int_{\Omega^1} 1\). It follows from the definition of \(\pi^*\) that
\[
(z^* + au^*, P_h u^* - u^*) \\
= (z^* + au^*, P_h u^* - u^*)|_{\Omega^0} \\
\leq (z^* + au^* - \pi^*(z^* + au^*), P_h u^* - u^*)|_{\Omega^0} \\
\leq Ch^2 \|z^* + au^*\|_{\Omega^1} \|u^*\|_{\Omega^1} \\
\leq Ch^2 \|z^* + au^*\|_{\Omega^1,\infty} \|u^*\|_{\Omega^1,\infty} \|\Omega^1\| \\
\leq Ch^3. \quad (50)
\]

Then it follows from assumption (41), (45)–(50), and the Schwartz inequality that
\[
c \|P_h u^* - u^*\|^2_0 \\
\leq (J'(P_h u^*) - J'(u^*), P_h u^* - u^*) \\
= \beta (P_h u^* - u^*, P_h u^* - u^*) \\
= -(z^* - z_h^*)(P_h u^*), P_h u^* - u^* \\
\leq Ch^3 + Ch^2 \|P_h u^* - u^*\|^2 \\
\leq Ch^3 + \delta \|P_h u^* - u^*\|^2. \quad (51)
\]

The estimate (42) follows from taking \(\delta = \frac{1}{2}c\). \(\square\)

Next, we establish the following superconvergence result for state and co-state.

**Theorem 2** Suppose that assumptions (A1–3) are satisfied. Let \((p^*, y^*, q^*, z^*, u^*) \in (V \times W)^2 \times U_{ad}\) be the solutions defined in (8)–(12) and \((p_h^*, y_h^*, q_h^*, z_h^*, u_h^*) \in (V_h \times W_h)^2 \times U_h\) be the solutions of (22)–(26). Then we have
\[
\begin{align*}
\|\Pi_k p_h^* - p_h^*\|_{\text{div}} + \|P_h y^* - y_h^*\| & \leq Ch^{3/2}, \quad (52) \\
\|\Pi_k q_h^* - q_h^*\|_{\text{div}} + \|P_h z^* - z_h^*\| & \leq Ch^{3/2}. \quad (53)
\end{align*}
\]

Proof: It follows from (8)–(12) and (22)–(26) that we have the error equations:
\[
\begin{align*}
(A^{-1}(p^* - p_h^*), v_h) - (y^* - y_h^*, \text{div} v_h) &= 0, \\
(\text{div} (p^* - p_h^*), w_h) + (\phi'(y^*)(y^* - y_h^*), w_h) &= (u^* - u_h^*, w_h), \\
(A^{-1}(q^* - q_h^*), v_h) - (z^* - z_h^*, \text{div} v_h) &= -(p^* - p_h^*, v_h), \\
(\text{div} (q^* - q_h^*), w_h) + (\phi'(y^*)(z^* - z_h^*), w_h) &= (y^* - y_h^*, w_h) - (\phi''(y^*)(y^* - y_h^*)z_h^*, w_h),
\end{align*}
\]

for all \(v_h \in V_h\) and \(w_h \in W_h\). By using the definitions of projections \(\Pi_h\) and \(P_h\), the above equations can be rewritten as
\[
\begin{align*}
(A^{-1}(\Pi_h p^* - p_h^*), v_h) - (P_h y^* - y_h^*, \text{div} v_h) &= \phi_1(v_h), \quad (54) \\
(\text{div} (\Pi_h p^* - p_h^*), w_h) + (\phi'(y^*)(P_h y^* - y_h^*), w_h) &= \psi_1(w_h), \quad (55) \\
(A^{-1}(\Pi_h q^* - q_h^*), v_h) - (P_h z^* - z_h^*, \text{div} v_h) &= \phi_2(v_h), \quad (56) \\
(\text{div} (\Pi_h q^* - q_h^*), w_h) + (\phi'(y^*)(P_h z^* - z_h^*), w_h) &= \psi_2(w_h), \quad (57)
\end{align*}
\]

for all \(v_h \in V_h\) and \(w_h \in W_h\), where
\[
\begin{align*}
\phi_1(v_h) &= -(A^{-1}(p^* - \Pi_h p^*), v_h), \\
\psi_1(w_h) &= (u^* - u_h^*, w_h) - (\phi'(y^*)(y^* - P_h y^*), w_h), \\
\phi_2(v_h) &= -(p^* - p_h^*, v_h) - (A^{-1}(q^* - \Pi_h q^*), v_h), \\
\psi_2(w_h) &= (y^* - y_h^*, w_h) - (\phi'(y^*)(z^* - P_h z^*), w_h)
\end{align*}
\]

since the terms \(\phi_1(v_h), \psi_1(w_h), \phi_2(v_h), \psi_2(w_h)\) can be regarded as linear functionals of \(v_h\) and \(w_h\) defined on \(V_h\) and \(W_h\), respectively. Then we know from the stability result\(^{19,20}\) that
\[
\begin{align*}
\|\Pi_k p_h^* - p_h^*\|_{\text{div}} + \|P_h y^* - y_h^*\| & \leq c \left\{ \sup_{v_h \in V_h} \|\phi_1(v_h)\|_{\text{div}} + \sup_{w_h \in W_h} \|\psi_1(w_h)\| \right\}, \\
\|\Pi_k q_h^* - q_h^*\|_{\text{div}} + \|P_h z^* - z_h^*\| & \leq c \left\{ \sup_{v_h \in V_h} \|\phi_2(v_h)\|_{\text{div}} + \sup_{w_h \in W_h} \|\psi_2(w_h)\| \right\}.
\end{align*}
\]

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It is easy to see that

\[ (p^* - p_h^*)\nu_h = (p^* - \Pi_h p^*, \nu_h) + (\Pi_h p^* - p_h^*, \nu_h), \]

\[ (y^* - y_h^*, w_h) = (y^* - P_h y^*, w_h) + (P_h y^* - y_h^*, w_h) \]

By the standard superconvergence of mixed finite element methods\(^{21-23}\), we have

\[ (\phi')(y^*) (y^* - P_h y^*, w_h) \leq C h^2 \| y^* \|_{H^1(\Omega)} \| w_h \|. \]

\[ (\phi')(y^*) (z^* - P_h z^*, w_h) \leq C h^2 \| z^* \|_{H^1(\Omega)} \| w_h \|. \]

\[ (\phi'')(y^*) (y^* - P_h y^*, z_h^*, w_h) \leq C h^2 \| y^* \|_{H^1(\Omega)} \| w_h \|. \]

Here we only give the proof of (60). By using the definition of the local \(L^2(\Omega)\) projection \(P_h\), we obtain

\[ (P_h (\phi')(y^*)) (y^* - P_h y^*, w_h) = (y^* - P_h y^*, P_h (\phi')(y^*)) w_h = 0. \]

Then

\[ (\phi')(y^*) (y^* - P_h y^*, w_h) = (\phi'(y^*) - P_h \phi' (y^*)) (y^* - P_h y^*, w_h) \leq C h \| \phi \|_{L^2(\Omega)} \| y^* - P_h y^* \| \| w_h \| \]

\[ \leq C h^2 \| y^* \|_{H^1(\Omega)} \| w_h \|. \]

Under the condition \( y^*, z^* \in H^2(\Omega) \), applying the integral identity technique\(^{24}\), we see that

\[ (A^{-1} (p^* - \Pi_h p^*), \nu_h) \leq C h^2 \| y^* \|_{H^1(\Omega)} \| \nu_h \| , \]

\[ (A^{-1} (q^* - \Pi_h q^*), \nu_h) \leq C h^2 \| z^* \|_{H^1(\Omega)} \| \nu_h \|, \]

\[ (p^* - \Pi_h p^*, \nu_h) \leq C h^2 \| y^* \|_{H^1(\Omega)} \| \nu_h \|. \]

By adding (54) and (55) to \( \nu_h = \Pi_h p^* - p^*_h \) and \( w_h = P_h y^* - y^*_h \), we have

\[ (A^{-1} (\Pi_h p^* - p^*_h), \Pi_h p^* - p^*_h) \]

\[ + (\phi'(y^*) (P_h y^* - y^*_h), P_h y^* - y^*_h) \]

\[ = (u^* - u_h^*, P_h y^* - y^*_h) \]

\[ - (\phi'(y^*) (y^* - P_h y^*), P_h y^* - y^*_h) \]

\[ - (A^{-1} (p^* - \Pi_h p^*), \Pi_h p^* - p^*_h). \]

Using the assumption of \( A(x) \), \( \phi \) and Hölder’s inequality, for any small \( \delta > 0 \), we obtain

\[ \frac{1}{C^2} \| \Pi_h p^* - p^*_h \|^2 + \lambda \| P_h y^* - y^*_h \|^2 \leq (A^{-1} (\Pi_h p^* - p^*_h), \Pi_h p^* - p^*_h) \]

\[ + (\phi'(y^*) (P_h y^* - y^*_h), P_h y^* - y^*_h) \]

\[ = (u^* - u_h^*, P_h y^* - y^*_h) \]

\[ - (\phi'(y^*) (y^* - P_h y^*), P_h y^* - y^*_h) \]

\[ - (A^{-1} (p^* - \Pi_h p^*), \Pi_h p^* - p^*_h) \]

\[ \leq \| u^* - u_h^* \| \cdot \| P_h y^* - y^*_h \|

\[ + \| \phi \|_{L^2(\Omega)} \| y^* - P_h y^* \| \cdot \| P_h y^* - y^*_h \|

\[ + C \| p^* - \Pi_h p^* \| \cdot \| \Pi_h p^* - p^*_h \|

\[ \leq \| u^* - u_h^* \| \cdot \| P_h y^* - y^*_h \|

\[ + Ch^2 \| y^* \|_{H^1(\Omega)} \cdot \| P_h y^* - y^*_h \|

\[ + Ch^2 \| y^* \|_{H^1(\Omega)} \cdot \| \Pi_h p^* - p^*_h \|

\[ \leq Ch^4 + C \| u^* - u_h^* \| ^2

\[ + \delta (\| P_h y^* - y^*_h \|^2 + \| \Pi_h p^* - p^*_h \|^2). \]

The formula is equivalent to

\[ \| \Pi_h p^* - p^*_h \| + \| P_h y^* - y^*_h \| \leq Ch^2 + C \| u^* - u_h^* \|. \]

Note that

\[ (u^* - u_h^*, w_h) = (u^* - P_h u^*, w_h) + (P_h u^* - u_h^*, w_h). \]

It is easy to see that \( (u^* - P_h u^*, w_h) = 0 \). By using Theorem 1, we clearly see that

\[ (P_h u^* - u_h^*, w_h) \leq \| P_h u^* - u_h^* \| \cdot \| w_h \|

\[ \leq Ch^{3/2} \| w_h \|. \]

From the above analysis, we can obtain (52) and (53).

**NUMERICAL EXAMPLES**

We present two examples to test the superconvergence results of the control. The optimization problems were solved numerical by projected
gradient methods, with codes developed based on AFEPACK. The control function \( u \) is discretized by piecwise constant functions, where the state \((y, p)\) and the co-state \((z, q)\) were approximated by the lowest order Raviart-Thomas mixed finite element functions. In the two examples, we choose the domain \( \Omega = [0, 1] \times [0, 1] \).

**Example 1** We consider the two-dimensional nonlinear elliptic optimal control problem

\[
\min_{u \in U_{ad}} \frac{1}{2} \{ \| p - p_d \|^2 + \| y - y_d \|^2 + \| u \|^2 \}
\]

subject to the state equation

\[
\text{div} \ p + y^3 = u + f, \quad p = -\nabla y, \quad x \in \Omega,
\]

with the boundary condition \( y = 0, \ x \in \partial \Omega \), and the admissible set \( U_{ad} = \{ u \in L^2(\Omega) : \ u \geq 0 \} \). Next, we introduce the co-state elliptic equation \( \text{div} \ q + 3y^2z = y - y_d, \ q = -(\nabla z + p - p_d), \ x \in \Omega \), with the boundary condition \( z = 0, \ x \in \partial \Omega \). We choose that

\[
u = \max(0, -z),
\]

\[
f = 8\pi^2 y + y^3 - u,
\]

\[
y_d = (1 - 16\pi^2)y - 3y^2z,
\]

\[
y = \sin(2\pi x_1)\sin(2\pi x_2),
\]

\[
z = 2\sin(2\pi x_1)\sin(2\pi x_2),
\]

\[
p = -2\pi\cos 2\pi x_1\sin 2\pi x_2\cos 2\pi x_2, \cos 2\pi x_1\sin 2\pi x_1,
\]

\[
q = -\pi\cos 2\pi x_1\sin 2\pi x_2\cos 2\pi x_2\sin 2\pi x_1,
\]

\[
p_d = \pi\cos 2\pi x_1\sin 2\pi x_2\cos 2\pi x_2\sin 2\pi x_1.
\]

In the numerical implementation, the profile of the numerical solution is shown in Fig. 1 and the errors \( \| u^* - u_h^* \| \) and \( \| P_h u^* - u_h^* \| \) obtained on a sequence of uniformly refined meshes are presented in Table 1.

**Example 2** We consider the following nonlinear optimal control problem:

\[
\min_{u \in U_{ad}} \frac{1}{2} \{ \| p - p_d \|^2 + \| y - y_d \|^2 + \| u \|^2 \},
\]

\[
\text{div} \ p + y^7 = u + f, \quad p = -\nabla y, \quad x \in \Omega,
\]

\[
y = 0, \quad x \in \partial \Omega,
\]

and we introduce co-state elliptic equation \( \text{div} \ q + 7y^6z = y - y_d, \ q = -(\nabla z + p - p_d), \ x \in \Omega \), with the boundary condition \( z = 0, \ x \in \partial \Omega \). We choose that

\[
u = \max(-z, 0),
\]

\[
f = \text{div} \ p + y^7 - u,
\]

\[
y_d = y - \text{div} \ q - 7y^6z, \quad q = -\nabla z - p + p_d,
\]

\[
y = 2x_1x_2(1 - x_1^3)(1 - x_2^3)\sin(8\pi x_1),
\]

\[
z = -x_1x_2(1 - x_1^3)(1 - x_2^3)\sin(8\pi x_1),
\]

\[
p = -\nabla y, \quad p_d = p + q + \nabla z.
\]

The profile of the numerical solution is presented in Fig. 3. The superconvergence behaviour of \( \| P_h u^* - u_h^* \| \) is illustrated in Table 2 and Fig. 4.
Fig. 3 The profile of the numerical solution of Example 2.

Fig. 4 Convergence orders of $u^* - u_h^*$ and $P_hu^* - u_h^*$.

From the numerical results of the examples, the superconvergence phenomenon can be observed clearly.

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REFERENCES