Reverses and variations of the Young inequality

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ABSTRACT: We extend the range of the weighted operator means for \( v \notin [0, 1] \) and obtain some corresponding operator inequalities. We also present several reversed Young-type inequalities.

KEYWORDS: weighted operator, positive operator, binary operation, Hilbert-Schmidt norm, Young-type inequality

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INTRODUCTION

Let \( B(H) \) be the \( C^* \)-algebra of all bounded linear operators on a Hilbert space \( H \) equipped with the operator norm, \( S(H) \) the set of all bounded self-adjoint operators, and \( \mathbb{P} = \mathbb{P}(H) \) the open convex cone of all positive invertible operators. For \( X, Y \in S(H) \), we write \( X \preceq Y \) if \( Y - X \) is positive, and \( X < Y \) if \( Y - X \) is positive invertible.

The unitarily invariant norm \( \| \cdot \| \) is defined on the matrix algebra \( M_n \) of all \( n \times n \) matrices with entries in the complex field \( \mathbb{C} \). For \( A = (a_{ij}) \in M_n \), the Hilbert-Schmidt norm of \( A \) is defined by \( \| A \|_2 = (\sum_{i=1}^{n} s_i^2(A))^{1/2} \), where \( s_1(A), s_2(A), \ldots, s_n(A) \) are the singular values of \( A \), i.e., the eigenvalues of the positive matrix \( |A| = (A^*A)^{1/2} \) where \( A^* = (A^T) \), arranged in decreasing order and repeated according to multiplicity. It is known that the Hilbert-Schmidt norm is unitarily invariant.

Let \( a, b > 0 \) be two positive real numbers and \( v \in [0, 1] \). The \( v \)-weighted arithmetic and geometric means of \( a \) and \( b \), denoted by \( A_v(a, b) \) and \( G_v(a, b) \), respectively, are defined as

\[
A_v(a, b) = (1-v)a + vb, \quad G_v(a, b) = a^{1-v}b^v.
\]

Note that \( A_v(a, b) \geq G_v(a, b) \) for all \( v \in [0, 1] \). This is the well-known Young inequality. In particular, if \( v = \frac{1}{2} \) then \( A_{1/2}(a, b) = \frac{1}{2}(a + b) \) and \( G_{1/2}(a, b) = \sqrt{ab} \) are the arithmetic and geometric means, respectively. The Heinz mean of \( a \) and \( b \) is defined as

\[
H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}
\]

for \( v \in [0, 1] \). For \( v = 0, 1 \), this is equal to arithmetic mean and for \( v = \frac{1}{2} \) it is the geometric mean.

Let \( A, B \in B(H) \) be two positive operators and \( v \in [0, 1] \). The \( v \)-weighted arithmetic mean of \( A \) and \( B \), denoted by \( A_{\nabla} v B \), is defined as

\[
A_{\nabla} v B = (1-v)A + vB.
\]

If \( A \) is invertible, the \( v \)-weighted geometric mean of \( A \) and \( B \), denoted by \( A_{\ast} v B \), is defined as

\[
A_{\ast} v B = A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}.
\]

For more details, see Ref. 1. When \( v = \frac{1}{2} \), we write \( A_{\nabla} B \) and \( A_{\ast} B \) for brevity, respectively.

The operator version of the Heinz mean, denoted by \( H_v(A, B) \), is defined as

\[
H_v(A, B) = \frac{A_{\nabla} B + A_{\ast} B}{2}, \quad 0 \leq v \leq 1.
\]

It is well known that if \( A \) and \( B \) are positive invertible operators, then

\[
A_{\nabla} B \geq A_{\ast} B, \quad 0 \leq v \leq 1.
\]

The Specht ratio\(^2\) is defined by

\[
S(t) = \frac{t^{1/(t-1)}}{\log t^{1/(t-1)}} \quad \text{for} \quad t > 0, t \neq 1,
\]

and

\[
S(1) = \lim_{t \to 1} S(t) = 1.
\]

Furuichi\(^4\) gave the following refined version:

\[
A_{\nabla} v B \geq S(h^r) A_{\ast} v B \geq A_{\ast} v B,
\]

where \( r = \min\{v, 1-v\} \). Zuo et al\(^5\) gave another one:

\[
K(h, 2)^r A_{\ast} v B \leq A_{\nabla} v B,
\]
where $K(t, 2) = (t + 1)^2/4t$ for $t > 0$ is the Kantorovich constant. In Ref. 6, Furuichi gave another refined version:

$$A\nabla v B \geq A\nabla v B + 2r(A\nabla v B - A\nabla B) \geq A\nabla B.$$ 

Recently there have been a number of other studies on similar topics and various improvement versions.\(^7\)\(^-\)\(^11\)

The Heinz norm inequality, which is one of the essential inequalities in operator theory, states that for any positive operators $A, B \in M_n$, any operator $X \in M_n$, and $v \in [0, 1]$, the following double inequality holds:

$$2\|A^{1/2}XB^{1/2}\| \leq \|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leq \|AX + XB\|.$$ \hspace{1cm} (1)

Kittaneh and Manasrah\(^1\)\(^2\) showed a refined version of the right-hand side of inequality (1) for the Hilbert-Schmidt norm as follows:

$$\|A^{1/2}XB^{1/2}\| \leq \|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leq \|AX + XB\|,$$ \hspace{1cm} (2)

where $A, B, X \in M_n$ such that $A, B$ are positive semidefinite, $v \in [0, 1]$ and $r_0 = \min\{v, 1-v\}$. Kaur et al\(^1\)\(^3\) using the convexity of the function $f(v) = \|A^{1/2}XB^{1/2}\|$, with $v \in [0, 1]$, presented more refinements of the Heinz inequality.

It was shown in Ref. 14 that a reverse of inequality (2) is

$$\|AX + XB\| \leq \|A^{1/2}XB^{1/2}\| \leq \|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leq \|AX + XB\|,$$ \hspace{1cm} (3)

where $A, B, X \in M_n$ such that $A, B$ are positive semidefinite, $v \in [0, 1]$, and $r_0 = \max\{v, 1-v\}$.

In this paper, we extend the range of the definition of the weighted operator means for $v \notin [0, 1]$ and obtain some corresponding operator inequalities. We also present a reverse of (2) and some other operator inequalities.

**SOME OPERATOR INEQUALITIES FOR $v \notin [0, 1]$**

For $A, B \in \mathbb{P}$ and $v \in [0, 1]$, the $v$-weighted geometric operator mean is defined as:

$$A\nu v B = A^{1/2}(A^{-1/2}BA^{-1/2})^vA^{1/2}.$$ 

For convenience, we use the notation $\tau_v$ and $H_v$ for the binary operation $A\nu v B = A^{1/2}(A^{-1/2}BA^{-1/2})^vA^{1/2}$, and let $f$ and $g$ be continuous real functions such that $f(t) \geq g(t)$ for all $t \in \mathbb{R}$. Hence, for $x \geq 0$,\(^{15}\)

$$\left(\int f(x)dx\right)^2 \leq \int (f(x))^2dx \leq \left(\int g(x)dx\right)^2.$$ \hspace{1cm} (i)

(ii) $a + (v - 1)\frac{(\sqrt{a} - \sqrt{b})^2}{2} \leq a^{1-v}b^v$.

(iii) $(a + b)^2 + 2(v - 1)(\sqrt{a} - \sqrt{b})^2 \leq a^{1-v}b^v + b^{1-v}a^v$.

Proof: Let $a, b > 0$ and $v \notin [0, 1]$. (i) Assume that $f(t) = t^{1-v} - v + (v - 1)t$ with $t \in (0, \infty)$. It is easy to see that $f(t)$ has a minimum at $t = 1$ in the interval $(0, \infty)$. Hence $f(t) \geq f(1) = 0$ for all $t > 0$. Assume that $a, b > 0$. Letting $t = b/a$, we get:

$$va + (1-v)b \leq a^{1-v}b^v.$$ 

So we have

$$va + (1-v)b + (v - 1)(\sqrt{a} - \sqrt{b})^2 = (2 - 2v)\sqrt{ab} + (2v - 1)a \leq (\sqrt{ab})^{2-2v}a^{2v-1} = a^{1-v}b^v.$$ 

(ii) It can be proved in a similar fashion to (i). (iii) It follows from (ii) by replacing $a$ by $a^2$ and $b$ by $b^2$.\(\square\)
\textbf{Theorem 1} Let $A, B \in \mathbb{P}$ and $v \notin [0, 1]$. Then:
\[ vA + (1-v)B + 2(v-1)(A\nabla B - A\nabla B) \leq A_{t_{1-v}}B. \]

\textit{Proof:} By Lemma 2(i), we have
\[ v + (1-v)b + (v-1)(1-\sqrt{b})^2 \leq b^{1-v}, \]
for any $b > 0$. If $X = A^{-1/2}BA^{-1/2}$ and thus $\text{Sp}(X) \subseteq (0, +\infty)$, then we have
\[ v + (1-v)t + (v-1)(1-\sqrt{t})^2 \leq t^{1-v}, \]
for any $t \in \text{Sp}(X)$. This is the same as
\[ vI + (1-v)X + (v-1)(I - X^{1/2})^2 \leq X^{1-v}. \quad (4) \]
Multiplying both sides of (4) by $A^{1/2}$, we get
\[ vA + (1-v)B + (v-1)(A+B - 2A^{1/2}X^{1/2}A^{1/2}) \leq A^{1/2}X^{1-v}A^{1/2}. \quad (5) \]
If $v \notin [0, 1]$, then
\[ vA + (1-v)B + 2(v-1)(A\nabla B - A\nabla B) \leq A_{t_{1-v}}B. \]
\[ \square \]

\textbf{Remark 1} In Ref. 12, the authors showed that if $v \in (0, \frac{1}{2})$, then
\[ vA + (1-v)B + 2(v-1)(A\nabla B - A\nabla B) \leq A_{t_{1-v}}B. \]
It is the same version of the formula (5). Hence for all $v \notin \left[\frac{1}{2}, 1\right]$,\[ vA + (1-v)B + 2(v-1)(A\nabla B - A\nabla B) \leq A_{t_{1-v}}B \]
holds.

\textbf{Remark 2} If $A, B \in \mathbb{P}$ and $B \succ A$, $v \in (1, 2)$, then by the monotonicity of $A_t$, and $0 < v - 1 < 1$, $B^{-1} \leq A^{-1},$
\[ vA + (1-v)B + 2(v-1)(A\nabla B - A\nabla B) \leq A_{t_{1-v}}B \]
\[ = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-v}A^{1/2} \]
\[ = A^{1/2}(A^{1/2}B^{-1}A^{1/2})^{1-v}A^{1/2} \]
\[ \leq A^{1/2}(A^{1/2}A^{-1}A^{1/2})^{1-v}A^{1/2} = A. \]
This is the same as
\[ 0 \leq A\nabla B - A\nabla B \leq \frac{B-A}{2}. \]
By Lemma 2 (ii), (iii) and using the same processing technique as in Theorem 1, we can get the following theorems and the corresponding remarks.

\textbf{Theorem 2} Let $A, B \in \mathbb{P}$ and $v \notin [0, 1]$. Then
\[ A\nabla B + 2(v-1)(A\nabla B - A\nabla B) \leq H^2(A, B). \]

\textbf{Remark 3} In Ref. 14, the authors showed that if $v \in (0, \frac{1}{2})$, then
\[ A\nabla B + 2(v-1)(A\nabla B - A\nabla B) \leq H^2(A, B). \]
Hence for all $v \notin \left[\frac{1}{2}, 1\right],$
\[ A\nabla B + 2(v-1)(A\nabla B - A\nabla B) \leq H^2(A, B) \]
holds.

\textbf{Remark 4} If $A, B \in \mathbb{P}$ and $B \succ A$, $v \in (1, 2)$, then
\[ B + 4(v-1)(A\nabla B - A\nabla B) \leq A_{t_v}B. \]

\textbf{Theorem 3} Let $A, B \in \mathbb{P}$ and $v \notin [0, 1]$. Then
\[ (2v-1)(A + A_{t_v}B) - 4(v-1)B \leq A_{t_{2-v}}B + A_{t_{2-v}}B. \]

\textbf{Remark 5} If $A, B \in \mathbb{P}$ and $B \succ A$, $v \in (1, 2)$, then
\[ 2(v-1)(A - 2B) + (2v-1)A_{t_v}B \leq A_{t_{2-v}}B. \]

\section*{A REVERSE OF THE HEINZ INEQUALITY FOR MATRICES}

In this section, we present a reverse of the Heinz inequality for matrices. To obtain the result, we need the following lemma.

\textbf{Lemma 3} (Ref. 17) Let $a, b > 0$. If $0 \leq v \leq \frac{1}{2}$, then
\[ v^2a + (1-v)^2b \leq (1-v)^2(\sqrt{a} - \sqrt{b})^2 \]
\[ + a^v[(1-v)^2b]^{1-v}. \quad (6) \]
If $\frac{1}{2} \leq v \leq 1$, then
\[ v^2a + (1-v)^2b \leq v^2(\sqrt{a} - \sqrt{b})^2 + (v^2a)^v b^{1-v}. \quad (7) \]

Based on Lemma 3, the following corollaries can be easily obtained.

\textbf{Corollary 1} Let $a, b > 0$. If $0 \leq v \leq \frac{1}{2}$, then
\[ 2v(a + b) \leq 2(1-v)(\sqrt{a} - \sqrt{b})^2 \]
\[ + (1-v)^{1-2v}[a^v b^{1-v} + b^v a^{1-v}]. \quad (8) \]
If $\frac{1}{2} \leq v \leq 1$, then
\[ 2(1-v)(a + b) \leq 2v(\sqrt{a} - \sqrt{b})^2 \]
\[ + v^{2v-1}[a^v b^{1-v} + b^v a^{1-v}]. \quad (9) \]
Corollary 2 Let $a$, $b > 0$. If $0 \leq \nu \leq \frac{1}{2}$, then
\[
2\nu(a + b)^2 \leq 2(1 - \nu)(a - b)^2 + (1 - \nu)^{1 - 2\nu}(a^\nu b^{1 - \nu} + b^\nu a^{1 - \nu})^2. \tag{10}
\]
If $\frac{1}{2} \leq \nu \leq 1$, then
\[
2(1 - \nu)(a + b)^2 \leq 2\nu(a - b)^2 + \nu^{2\nu - 1}(a^\nu b^{1 - \nu} + b^\nu a^{1 - \nu})^2. \tag{11}
\]

Theorem 4 Let $A, B, X \in \mathbb{M}_n$ with $A, B$ are positive, and $\nu \in [0, 1]$. Then
\[
2\nu\|AX + XB\|_2^2 \leq 2(1 - \nu)\|AX - XB\|_2^2 + (1 - \nu)^{1 - 2\nu}\|A^\nu XB^{1 - \nu} + A^{1 - \nu}XB^\nu\|_2^2
\]
for $0 \leq \nu \leq \frac{1}{2}$, and
\[
2(1 - \nu)\|AX + XB\|_2^2 \leq 2\nu\|AX - XB\|_2^2 + \nu^{2\nu - 1}\|A^\nu XB^{1 - \nu} + A^{1 - \nu}XB^\nu\|_2^2
\]
for $\frac{1}{2} \leq \nu \leq 1$.

Proof: By spectral decomposition, there are unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U\Lambda_1U^*$ and $B = V\Lambda_2V^*$, where
\[
\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)
\]
and
\[
\Lambda_2 = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)
\]
where $\lambda_i$ and $\mu_i$ for $i = 1, 2, \ldots, n$ are the eigenvalues of $A$ and $B$, respectively. Let $Y = U^*XV = [y_{ij}]$, then
\[
AX + XB = U[(\lambda_i + \mu_j)y_{ij}]V^*,
\]
\[
AX - XB = U[(\lambda_i - \mu_j)y_{ij}]V^*,
\]
\[
A^\nu XB^{1 - \nu} + A^{1 - \nu}XB^\nu
\]
\[
= U\Lambda_1^\nu U^*XV\Lambda_2^{1 - \nu}V^* + U\Lambda_1^{1 - \nu}U^*XV\Lambda_2^\nuV^*
\]
\[
= U\Lambda_1^\nu Y\Lambda_2^{1 - \nu}V^* + U\Lambda_1^{1 - \nu}Y\Lambda_2^\nuV^*
\]
\[
= U\left[\lambda_i^\nu\mu_j^{1 - \nu} + \lambda_i^{1 - \nu}\mu_j^\nu\right]y_{ij}V^*.
\]
If $0 \leq \nu \leq \frac{1}{2}$, then by (10) and the unitary invariance of the Hilbert-Schmidt norm, we have
\[
2\nu\|AX + XB\|_2^2 = 2\nu\sum_{i,j=1}^n(\lambda_i + \mu_j)^2|y_{ij}|^2
\]
\[
\leq 2(1 - \nu)\sum_{i,j=1}^n(\lambda_i - \mu_j)^2|y_{ij}|^2
\]
\[
+ (1 - \nu)^{1 - 2\nu}\sum_{i,j=1}^n(\lambda_i^{1 - \nu}\mu_j^\nu + \lambda_i^\nu\mu_j^{1 - \nu})^2|y_{ij}|^2
\]
\[
= 2(1 - \nu)\|AX - XB\|_2^2 + (1 - \nu)^{1 - 2\nu}\|A^\nu XB^{1 - \nu} + A^{1 - \nu}XB^\nu\|_2^2.
\]
If $\frac{1}{2} \leq \nu \leq 1$, then by (11) and using the same technique in the first part we get the other result. □

SOME REVERSES OF THE YOUNG-TYPE INEQUALITY FOR OPERATORS

In this section, we obtain some reverses of the Young-type inequality for two positive invertible operators.

Theorem 5 Let $A, B \in \mathcal{P}$ and $\nu \in [0, 1]$. Then
\[
\nu^2A + (1 - \nu)^2B \leq (2\nu - 1)^2(A\nabla B - A\sharp B)
\]
\[
+ (1 - \nu)^{2(1 - \nu)}A^\sharp_{1 - \nu}B,
\]
for $0 \leq \nu \leq \frac{1}{2}$, and
\[
\nu^2A + (1 - \nu)^2B \leq 2\nu^2(A\nabla B - A\sharp B) + \nu^{2\nu}A^\sharp_{1 - \nu}B,
\]
for $\frac{1}{2} \leq \nu \leq 1$.

Proof: For $0 \leq \nu \leq \frac{1}{2}$, by (6) we have
\[
\nu^2 a_{ij}(1 - \nu)^2 b_{ij} \leq (1 - \nu^2)(\sqrt{a_{ij}} - \sqrt{b_{ij}})^2 + (\nu^2 - 1)^2 b_{ij} \leq (1 - \nu)^2(1 - \sqrt{b_{ij}})^2 + [(1 - \nu)^2 b_{ij}]^{1 - \nu},
\]
for any $b > 0$. If $X = A^{-1/2}BA^{-1/2}$ and thus $\text{Sp}(X) \subseteq (0, +\infty)$, then we have
\[
\nu^2 + (1 - \nu)^2 b \leq (1 - \nu)^2(1 - \sqrt{b})^2 + [(1 - \nu)^2 b]^{1 - \nu},
\]
for any $t \in \text{Sp}(X)$. This is the same as
\[
\nu^2 t + (1 - \nu)^2 tX \leq (1 - \nu^2)(t - X^{1/2})^2 + [(1 - \nu^2 tX)^{1 - \nu}. \tag{12}
\]
Multiplying both sides of (12) by $A^{1/2}$, we get
\[
\nu^2A + (1 - \nu)^2B \leq 2(\nu - 1)^2(A\nabla B - A\sharp B)
\]
\[
+ (1 - \nu)^{2(1 - \nu)}A^\sharp_{1 - \nu}B.
\]
□
Theorem 6 Let \( A, B \in \mathbb{P} \) and \( v \in [0, 1] \). Then
\[ 2vA\nabla B \leq 2(1-v)(A\nabla B - A^\# B) + (1-v)^{1-2v}H_v(A, B), \]
for \( 0 \leq v \leq \frac{1}{2} \), and
\[ 2(1-v)A\nabla B \leq 2v(A\nabla B - A^\# B) + v^{2v-1}H_v(A, B), \]
for \( \frac{1}{2} \leq v \leq 1 \).

Proof: By Corollary 2 and the same processing technique as in Theorem 5, we can easily obtain the result.

REFERENCES