Some inequalities of Hermite-Hadamard type for \( m \)-harmonic-arithmetically convex functions

Bo-Yan Xi\(^a\), Feng Qi\(^b,^*\), Tian-Yu Zhang\(^a\)

\(^a\) College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China
\(^b\) Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China

\(^*\)Corresponding author, e-mail: qifeng618@hotmail.com

ABSTRACT: We introduce the notion of \( m \)-harmonic-arithmetically convex functions and establish some integral inequalities of Hermite-Hadamard type for these functions.

KEYWORDS: \( m \)-HA-convex function, Hermite-Hadamard type inequality

MSC2010: 26A51 26D15

INTRODUCTION

A function \( f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R} \) is said to be convex if

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0,1] \). The concept of \( m \)-convex functions was introduced as follows\(^1\).

**Definition 1** For \( f : [0, b] \to \mathbb{R} \) and \( m \in (0, 1] \), if

\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
\]

is valid for all \( x, y \in [0, b] \) and \( t \in [0,1] \), then we say that \( f \) is an \( m \)-convex function on \([0, b] \).

The following inequalities of Hadamard-type were established for the above kinds of convex functions.

**Theorem 1** (Ref. 2) Let \( f : \mathbb{R} = [0, \infty) \to \mathbb{R} \) be \( m \)-convex and \( m \in (0, 1] \). If \( f \in L_1([a, b]) \) for \( 0 \leq a < b < \infty \), then

\[
\frac{1}{b-a} \int_a^b f(x) \; dx \\
\leq \min \left\{ \frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2} \right\}.
\]

**Theorem 2** (Ref. 3) Let \( f : \mathbb{R} \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0, 1] \). If \( 0 < a < b < \infty \) and \( f \in L_1([a, b]) \), then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) + mf(x/m) \; dx \\
\leq \frac{m+1}{4} \left[ \frac{f(a)+f(b)}{2} + m\frac{f(a/m)+f(b/m)}{2} \right].
\]

For more information on notions of various convex functions and their Hermite-Hadamard type inequalities see Refs. 4–14 and references therein.

**m-HARMONIC-ARITHMETICALLY CONVEX FUNCTIONS**

We first define \( m \)-harmonic-arithmetically convex functions.

**Definition 2** Let \( f : (0, b] \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} \) and \( m \in (0, 1] \) be a constant. If

\[
f\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1} \leq tf(x) + m(1-t)f(y) \quad (1)
\]

for all \( x, y \in (0, b] \) and \( t \in [0,1] \), then \( f \) is said to be an \( m \)-harmonic-arithmetically convex (or \( m \)-HA-convex) function. If the inequality (1) is reversed, then \( f \) is said to be an \( m \)-harmonic-arithmetically concave (or \( m \)-HA-concave) function.

**Example 1** Let \( m \in (0, 1] \) and \( f(x) = x^{-r} \) for \( x \in \mathbb{R}_+ \).
$\mathbb{R}_+$ and $r > 0$. If $r \geq 1$, we have

\[
f \left( \frac{t}{x} + \frac{m(1-t)x}{y} \right) = \left[ \frac{t y + m(1-t)x}{(xy)^r} \right] \leq tf(x) + m(1-t)f(y) \quad (2)
\]

for all $x, y > 0$ and $t \in [0,1]$. If $0 < r \leq 1$, the inequality (2) is reversed. This implies that

(i) if $0 < r \leq 1$, the function $f$ is $m$-HA-convex on $\mathbb{R}_+$;

(ii) if $r \geq 1$, the function $f$ is $m$-HA-concave on $\mathbb{R}_+$.

**INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE**

We now present several integral inequalities of Hermite-Hadamard type for $m$-harmonic-arithmetic convex functions.

**Theorem 3** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $m \in (0,1]$ be a constant. If $f$ is an $m$-HA-convex function on $(0,b/m^2]$ and $f \in L_1([a,b/m])$ for $a, b \in \mathbb{R}_+$ with $a < b$, then

\[
f(H(a,b)) \leq \frac{1}{b-a} \int_a^b f(x) + m \left( \frac{b-a}{ab} x \right) \frac{1}{m-1} + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right),
\]

where $H(a,b) = 2ab/(a+b)$ is the well-known harmonic mean of two positive numbers $a$ and $b$,

\[
Q \left( \frac{u}{m_1}, \frac{v}{m_2} \right) = m_1 f \left( \frac{u}{m_1} \right) + m_2 f \left( \frac{v}{m_2} \right).
\]

and

\[
M(u,v) = \frac{v[(v-u)u - v(lnv-lnu)]}{(v-u)^2}
\]

for $v, u > 0$ with $v \neq u$ and $m_1, m_2 > 0$.

**Proof:** For $0 \leq t \leq 1$, by the $m$-HA convexity of $f$ on $(0,b/m^2]$, we obtain

\[
f(H(a,b)) = f \left( \frac{1}{b-a} \int_a^b f(x) \frac{1}{m-1} + m \left( \frac{b-a}{ab} x \right) \frac{1}{m-1} + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right) + M(b,a) Q \left( \frac{a}{m}, \frac{b}{m} \right),
\]

A straightforward computation gives

\[
\int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-2} dt = ab, \quad (6)
\]

\[
abM(b,a) = \int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-2} dt = \frac{a^2b[b(lnb-lna)-(b-a)]}{(b-a)^2}, \quad (7)
\]

\[
abM(a,b) = \int_0^1 \left( 1-t \right) \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-2} dt = \frac{ab^2[(b-a)-a(lnb-lna)]}{(b-a)^2}. \quad (8)
\]

Letting $x = (t/a + (1-t)/b)^{-1}$ for $0 \leq t \leq 1$ leads to

\[
\frac{ab}{b-a} \int_a^b f(x) dx = \int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-2} f \left( \left( \frac{t}{a} + \frac{1-t}{b} \right) \right) dt
\]

and using $x = ((1-t)/a + t/b)^{-1}$ for $0 \leq t \leq 1$ gives

\[
\frac{ab}{b-a} \int_a^b \left( \frac{b-a}{ab} x - 1 \right)^{-2} f \left( \frac{x}{m} \right) dx = \int_0^1 \frac{t}{a} + \frac{1-t}{b} \left( \frac{1-t}{ma} + \frac{t}{mb} \right)^{-1} dt. \quad (9)
\]

Multiplying both sides of the inequality (5) by $(t/a + (1-t)/b)^{-2}$ for $t \in [0,1]$, integrating with respect to $t \in [0,1]$, and using equations (6)–(9) we obtain the inequalities in (3).
Corollary 1 Under the conditions of Theorem 3, if \( m = 1 \) then

\[
\frac{f(H(a, b))}{b-a} \leq \frac{1}{b-a} \int_a^b \left[ \left( \frac{b-a}{ab} x - 1 \right)^{-1} + 1 \right] \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Theorem 4 Let \( f : \mathbb{R}_+ \to \mathbb{R}_0 \) and \( m \in (0, 1] \) be a constant. If \( f \) is an \( m \)-HA-convex function on \([0, b/m^2]\) and \( f \in L_1([a, b/m]) \) for \( a, b \in \mathbb{R}_+ \) with \( a < b \), then

\[
L(a, b)f(H(a, b)) \leq \frac{1}{b-a} \int_a^b \left[ \left( \frac{b-a}{ab} x - 1 \right)^{-1} + 1 \right] \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Proof: For \( 0 \leq t \leq 1 \), from (5), we have

\[
\int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt = abL(a, b),
\]

\[
abN(a, b) = \int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt = \frac{ab[(b-a)-a(ln b - ln a)]}{(b-a)^2},
\]

\[
abN(b, a) = \int_0^1 (1-t) \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt = \frac{ab[b(ln b - ln a)-(b-a)]}{(b-a)^2}.
\]

Taking \( x = (t/a + (1-t)/b) \) for \( 0 \leq t \leq 1 \) results in

\[
\int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} f\left( \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \right) \, dt = \frac{ab}{b-a} \int_a^b \frac{1}{x} f(x) \, dx
\]

and using \( x = ((1-t)/a+t/b)^{-1} \) for \( 0 \leq t \leq 1 \) gives

\[
\int_0^1 \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} f\left( \left( \frac{1-t}{ma} + \frac{t}{mb} \right)^{-1} \right) \, dt = \frac{ab}{b-a} \int_a^b \left( \frac{b-a}{ab} x^2 - x \right)^{-1} f\left( \frac{x}{m} \right) \, dx.
\]

Integrating both sides of the inequality (12) with respect to \( t \in [0, 1] \) and employing (13)–(16) produces the inequalities in (10).

Corollary 2 Under the conditions of Theorem 4, if \( m = 1 \), then

\[
L(a, b)f(H(a, b)) \leq \frac{1}{b-a} \int_a^b \left[ \left( \frac{b-a}{ab} x - 1 \right)^{-1} + 1 \right] f(x) \, dx \leq \frac{f(a) + f(b)}{2} L(a, b).
\]

Theorem 5 Let \( f : \mathbb{R}_+ \to \mathbb{R}_0 \) and \( m \in (0, 1] \) be a constant. If \( f \) is an \( m \)-HA-convex function on \([0, b/m^2]\) and \( f \in L_1([a, b/m]) \) for \( a, b \in \mathbb{R}_+ \) with \( a < b \), then

\[
2f(H(a, b)) \leq \frac{1}{b-a} \int_a^b \frac{ab}{x^2} \left[ f(x) + mf\left( \frac{x}{m} \right) \right] \, dx
\]
\[ \leq \frac{1}{2} \min \left\{ Q \left( \frac{a}{m}, \frac{b}{m^2} \right) + Q \left( \frac{a}{m}, \frac{b}{m} \right), Q \left( \frac{a}{m}, \frac{b}{m} \right) + Q \left( \frac{a}{m^2}, \frac{b}{m} \right), Q \left( \frac{a}{m}, \frac{b}{m^2} \right) + Q \left( \frac{a}{m}, \frac{b}{m^2} \right) \right\}, \]  

where \( Q(u, v) \) is defined as in (4).

**Proof**: Letting \( x = (t/a + (1-t)/b)^{-1} \) for \( 0 \leq t \leq 1 \) gives

\[
\frac{ab}{b-a} \int_a^b \frac{1}{x^2} f(x) \, dx = \int_0^1 f \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt
\]

and using \( x = ((1-t)/a + t/b)^{-1} \) for \( 0 \leq t \leq 1 \) gives

\[
\frac{ab}{b-a} \int_a^b \frac{1}{x^2} f \left( \frac{x}{m} \right) \, dx = \int_0^1 f \left( \frac{1-t}{ma} + \frac{t}{mb} \right)^{-1} \, dt.
\]

Integrating both sides of the inequality (5) with respect to \( t \in [0,1] \) and using (18) and (19) gives (17).

**Corollary 3** Under the conditions of Theorem 5, if \( m = 1 \), then

\[
\left( \frac{1}{a} - \frac{1}{b} \right) f(H(a, b)) \leq \int_a^b \frac{f(x)}{x^2} \, dx \leq \left( \frac{f(a) + f(b)}{2} \right)
\]

**Theorem 6** Let \( f : \mathbb{R}_+ \to \mathbb{R}_0 \) and \( m \in (0,1] \) be a constant. If \( f \) is an \( m \)-HA-convex function on \((0, b/m] \) and \( f \in L_1([a, b]) \) for \( a, b \in \mathbb{R}_+ \) with \( a < b \), then

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ f(a)M(b, a) + m f \left( \frac{b}{m} \right)M(a, b) \right\},
\]

\[
\frac{1}{b-a} \int_a^b \frac{1}{x} f(x) \, dx \leq \min \left\{ f(a)N(a, b) + m f \left( \frac{b}{m} \right)N(b, a) \right\},
\]

and

\[
\int_a^b \frac{f(x)}{x^2} \, dx \leq \left( \frac{f(a) + f(b)}{2} \right)
\]

where \( M(u, v) \) and \( N(u, v) \) are defined as in (4) and (11).

**Proof**: Putting \( x = (t/a + (1-t)/b)^{-1} \) for \( 0 \leq t \leq 1 \), using the \( m \)-HA-convexity of \( f \) on \((0, b/m) \), and using (7), (8), (14), and (15) yields

\[
\frac{ab}{b-a} \int_a^b f(x) \, dx = \int_0^1 f \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt
\]

and

\[
\frac{ab}{b-a} \int_a^b \frac{1}{x} f(x) \, dx = \int_0^1 f \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \, dt
\]

Integrating both sides of the inequality (5) with respect to \( t \in [0,1] \) and using (18) and (19) gives (17).

**Corollary 4** Under the conditions of Theorem 6, if \( m = 1 \), then

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq f(a)M(b, a) + f(b)M(a, b),
\]

and

\[
\frac{1}{b-a} \int_a^b \frac{1}{x} f(x) \, dx \leq f(a)N(a, b) + f(b)N(b, a),
\]

and

\[
\int_a^b f(x) \, dx \leq \left( \frac{f(a) + f(b)}{2} \right).
\]
\[
\int_a^b \frac{f(x)g(x)}{x^2} \, dx \leq \left( \frac{1}{a} \right) \left( 1 - \frac{1}{b} \right) \left( f(a)N(a, b) + f(b)N(b, a) \right),
\]

**Theorem 7** Let \( f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_0 \) and \( m \in (0, 1] \) be a constant. If \( f \) and \( g \) are \( m \)-HA-convex functions on \( (0, b/m) \) and \( f \) \( g \) \( \in L_1([a, b]) \) for \( a, b \in \mathbb{R}_+ \) with \( a < b \), then

\[
\int_a^b \frac{f(x)g(x)}{x^2} \, dx \leq \left( \frac{1}{a} \right) \left( 1 - \frac{1}{b} \right) \left[ 2f(a)g(a) + mf(a)g\left( \frac{b}{m} \right) + mf\left( \frac{b}{m} \right)g(a) + 2m^2 f\left( \frac{b}{m} \right)g\left( \frac{b}{m} \right) \right].
\]

**Proof:** Putting \( x = (t/a + (1-t)/b)^{-1} \) for \( 0 \leq t \leq 1 \) and using the \( m \)-HA-convexity of \( f \) and \( g \) on \( (0, b/m) \) yields

\[
\frac{ab}{b-a} \int_a^b \frac{1}{x^2} f(x)g(x) \, dx = \int_0^1 f\left( \frac{t}{a} \right) + \left( 1 - \frac{1}{b} \right) \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} g\left( \frac{t}{a} + \frac{1-t}{b} \right) \, dt \leq \int_0^1 \left[ tf(a) + m(1-t)g\left( \frac{b}{m} \right) \right] \, dt = \left[ f(a)g(a) + m(1-t)g\left( \frac{b}{m} \right) \right] \, dt = \left[ f(a)g(a) + mf\left( \frac{b}{m} \right)g\left( \frac{b}{m} \right) \right] \, dt + \left[ f(a)g\left( \frac{b}{m} \right) + mf\left( \frac{b}{m} \right)g(a) \right] \, dt = \frac{1}{6} \left[ 2f(a)g(a) + mf(a)g\left( \frac{b}{m} \right) + mf\left( \frac{b}{m} \right)g(a) + 2m^2 f\left( \frac{b}{m} \right)g\left( \frac{b}{m} \right) \right].
\]

**Corollary 5** Under the conditions of **Theorem 7**, if \( m = 1 \), then

\[
\int_a^b \frac{f(x)g(x)}{x^2} \, dx \leq \left( \frac{1}{a} \right) \left( 1 - \frac{1}{b} \right) \left[ 2f(a)g(a) + f(a)g(b) + f(b)g(a) + 2f(b)g(b) \right].
\]

**Acknowledgements:** This work was partially supported by the NNSF under Grant No. 11361038 of China and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant Nos. 2015MS0123 and 2014BS0106, China. The authors appreciate anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

**REFERENCES**

5. Li W-H, Qi F (2013) Some Hermite-Hadamard type inequalities for functions whose \( n \)-th derivatives are \((a, m)\)-convex. *Filomat* 27, 1575–82.