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Computation of a real eigenbasis for the Simpson discrete Fourier transform matrix

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ABSTRACT: In this paper, we demonstrate the usefulness of the duality property by using it to determine the spectrum of the Simpson discrete Fourier transform (SDFT) matrix of dimension $N \times N$, where $N \equiv 2 \pmod 4$, in finding an expression for the minimal polynomial. We determine the eigenvalues and their corresponding multiplicities. The SDFT matrix is diagonalizable. Thus there exists a basis for the underlying vector space consisting of eigenvectors. In light of this, we construct an eigenbasis for each subspace associated with each of the eight distinct eigenvalues.

KEYWORDS: eigenvalues

INTRODUCTION

It is well known that the discrete Fourier transform (DFT) can be obtained by a trapezoidal approximation of the integral used to approximate the Fourier coefficients of periodic functions[1]. This leads to the symmetric nature of the DFT transformation matrix. The DFT plays an important role in audio signal processing, adaptive filtering of artefacts from the human electroencephalogram and detection of fetal heartbeats from the fetal electrocardiogram[2]. We have proposed a DFT based on Simpson's numerical quadrature rule[1] and proved analogous properties to the classical DFT[3]. The eigenvalues of a matrix play an important role in the spectral resolution of functions of the matrix into their constituent components and the multiplicities give the number of linearly independent eigenvectors corresponding to each eigenvalue[4].

McClellan and Parks[5] studied the eigenstructure of the DFT matrix in detail and provided a means of numerically generating a linearly independent set of eigenvectors corresponding to the related eigenspaces. They made extensive use of the theory of Chebyshev sets[6] to prove the independence of the eigenvectors. We provide a numerical technique to generate a basis for the eigenspace corresponding to each of the eight distinct eigenvalues.

For any vector $\mathbf{f} = [f(0), f(1), \dots, f(N-1)]^{\mathrm{T}}$ of even length N, define the even indexed components by $\mathbf{f}_0 = [f(0), f(2), \dots, f(N-2)]^{\mathrm{T}}$, the odd indexed components by $\mathbf{f}_1 = [f(1), f(3), \dots, f(N-1)]^{\mathrm{T}}$ and the discrete transforms by

$$F_0(k) = \frac{2}{3}\sum_{j=0}^{N/2-1} \omega^{k(2j)} f_0(j),$$

$$F_1(k) = \frac{4}{3}\sum_{j=0}^{N/2-1} \omega^{k(2j+1)} f_1(j),$$

for $k = 0, 1, \dots, N-1$.

Define $\mathbf{F}_0 = [F_0(0), F_0(1), \dots, F_0(N-1)]^{\mathrm{T}}$ and $\mathbf{F}_1 = [F_1(0), F_1(1), \dots, F_1(N-1)]^{\mathrm{T}}$, then $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1$ represents the Simpson DFT transform of $\mathbf{f}$. The transformation can be written in matrix-vector notation as $\mathbf{F} = \mathbf{U}\mathbf{f}$, where

$$\mathbf{U} = \frac{2}{3}\begin{pmatrix} \mathbf{A} & 2\mathbf{DA} \\ \mathbf{A} & -2\mathbf{DA} \end{pmatrix}\mathbf{P},$$

the $(\frac{1}{2}N \times \frac{1}{2}N)$ matrix $\mathbf{A}$ is defined by its components

$$\mathbf{A}_{ij} = \omega^{2ij} \text{ for } i, j = 0, 1, \dots, \tfrac{N}{2} - 1,$$

the matrix $\mathbf{D} = \operatorname{diag}(1, \omega, \dots, \omega^{N/2-1})$ and the $(N \times N)$ permutation matrix $\mathbf{P}$ is defined by its components $P_{j+1,2j+1} = 1$ and $P_{N/2+j+1,2j+2} = 1$ for $j = 0, 1, \dots, \frac{1}{2}N - 1$. The effect of the permutation matrix is to separate the signal into even and odd indexed components, respectively, in accordance with the Simpson quadrature rule[1].

EIGENVALUES OF U

Using the duality property[7], we obtain an insight into the structure of the matrix $\mathbf{U}^4$. Exploiting this form, we obtain the spectrum $\sigma(\mathbf{U}^4)$. We state the duality property

$$\mathbf{U}^2\mathbf{f}(2k) = \tfrac{2}{3}Nf(-2k) - \tfrac{4}{9}Nf\left(-2k - \tfrac{N}{2}\right) \quad (1)$$

$$\begin{aligned}\mathbf{U}^2\mathbf{f}(2k+1) &= \tfrac{4}{3}Nf(-2k-1)\\ &\quad - \tfrac{2}{9}Nf\left(-2k-1-\tfrac{N}{2}\right). \quad (2)\end{aligned}$$

In order to apply the duality property to (1) and (2), we let $\mathbf{U}^2\mathbf{f}(2k) = g(2k)$ and $\mathbf{U}^2\mathbf{f}(2k+1) = g(2k+1)$ to obtain

$$\begin{aligned}\mathbf{U}^2\mathbf{g}(2k) &= \tfrac{2}{3}Ng(-2k) - \tfrac{4}{9}Ng\left(-2k-\tfrac{N}{2}\right)\\ &= \tfrac{2}{3}N\left[\tfrac{2}{3}Nf(2k) - \tfrac{4}{9}Nf\left(2k-\tfrac{N}{2}\right)\right]\\ &\quad - \tfrac{4}{9}N\left[\tfrac{4}{3}Nf\left(2k+\tfrac{N}{2}\right) - \tfrac{2}{9}Nf(2k)\right]\\ &= N^2\left[\tfrac{44}{81}f(2k) - \tfrac{8}{27}f\left(2k-\tfrac{N}{2}\right)\right]\\ &\quad - N^2\tfrac{16}{27}f\left(2k+\tfrac{N}{2}\right)\end{aligned}$$

and similarly it may be shown that

$$\begin{aligned}&\mathbf{U}^2\mathbf{g}(2k+1)\\ &= N^2\left[\tfrac{152}{81}f(2k+1) - \tfrac{8}{27}f\left(2k+1-\tfrac{N}{2}\right)\right]\\ &\quad - N^2\tfrac{4}{27}f\left(2k+1+\tfrac{N}{2}\right),\end{aligned}$$

using the periodicity N of $\mathbf{f}$, we get

$$\mathbf{U}^4\mathbf{f}(2k) = N^2\left[\tfrac{44}{81}f(2k) - \tfrac{8}{9}f\left(2k+\tfrac{N}{2}\right)\right] \quad (3)$$

and

$$\begin{aligned}\mathbf{U}^4\mathbf{f}(2k+1) &= N^2\tfrac{152}{81}f(2k+1)\\ &\quad - N^2\tfrac{4}{9}f\left(2k+1+\tfrac{N}{2}\right) \quad (4)\end{aligned}$$

for $k = 0, 1, \dots, \frac{1}{2}N - 1$.

Writing (3) and (4) in matrix-vector notation, we deduce that $\mathbf{U}^4$ is real and tridiagonal with the superdiagonal located in the first row in column $\frac{1}{2}N+1$ and the subdiagonal in the first column in row $\frac{1}{2}N+1$. The diagonal of matrix $\mathbf{U}^4$ has the form $\left[a, \frac{38}{11}a, a, \frac{38}{11}a, \dots, a, \frac{38}{11}a\right]$ of length N, the superdiagonal has the form $[-2b, -b, -2b, -b, \dots, -2b]$ of length $\frac{1}{2}N$ and a subdiagonal has the form $[-b, -2b, -b, -2b, \dots, -b]$ of length $\frac{1}{2}N$, where $a = \frac{44}{81}N^2$ and $b = \frac{4}{9}N^2$.

We now determine the eigenvalues of $\mathbf{U}^4$. Let $\mathbf{X} = [X(0), X(1), \dots, X(N-1)]$ be an eigenvector of $\mathbf{U}^4$ corresponding to the eigenvalue α then

$$\mathbf{U}^4\mathbf{X} = \alpha\mathbf{X}. \quad (5)$$

Let p be the index such that $|X(2k)| \leqslant |X(2p)|$ and m be the index such that $|X(2k+1)| \leqslant |X(2m+1)|$ for $k = 0, 1, \dots, \frac{1}{2}N - 1$, then it follows from (5) that

$$(a-\alpha)X(2p) - 2bX\left(\tfrac{N}{2}+2p\right) = 0, \quad (6)$$

$$-bX(2p) + \left(\tfrac{38}{11}a-\alpha\right)X\left(\tfrac{N}{2}+2p\right) = 0, \quad (7)$$

and

$$\left(\tfrac{38}{11}a-\alpha\right)X(2m+1) - bX\left(2m+1+\tfrac{N}{2}\right) = 0, \quad (8)$$

$$-2bX(2m+1) + (a-\alpha)X\left(2m+1+\tfrac{N}{2}\right) = 0. \quad (9)$$

If $X(2p) = X(2m+1) = 0$, then this contradicts the fact that $\mathbf{X}$ is an eigenvector. Suppose that $X(2p) \neq 0$, then if $X(\frac{1}{2}N+2p) = 0$, this contradicts (7); so $X(\frac{1}{2}N+2p) \neq 0$. Hence the linear system (6) and (7) has non-trivial solution; so the determinant

$$(a-\alpha)\left(\tfrac{38}{11}a-\alpha\right) + 2b^2 = 0. \quad (10)$$

If $X(2p) = 0$, then $X(2m+1) \neq 0$, and from equation (9), it follows that $X(2m+1+\frac{1}{2}N) \neq 0$, so that the linear system (8) and (9) has non-trivial solution, so that the determinant is zero which is the same as (10). The solution of the quadratic equation (10) provides the spectrum $\sigma(\mathbf{U}^4) = \{\frac{1}{81}(98+18\sqrt{17})N^2, \frac{1}{81}(98-18\sqrt{17})N^2\}$, from which it follows that

$$\sigma(\mathbf{U}) \subseteq \left\{r_1\,\mathrm{e}^{il\pi/2}, r_2\,\mathrm{e}^{il\pi/2} \mid l = 0,1,2,3\right\}, \quad (11)$$

where

$$r_1 = \frac{1}{3}\sqrt{N}\sqrt{9+\sqrt{17}}, \quad r_2 = \frac{1}{3}\sqrt{N}\sqrt{9-\sqrt{17}}.$$

THE MINIMAL POLYNOMIAL

Successively using the duality property, it can be shown that

$$\mathbf{U}^6\mathbf{f}(2k) = N^3\tfrac{136}{243}f(-2k) - N^3\tfrac{1040}{729}f\left(-2k+\tfrac{N}{2}\right),$$

$$\begin{aligned}\mathbf{U}^6\mathbf{f}(2k+1) &= N^3\tfrac{656}{243}f(-2k-1)\\ &\quad - N^3\tfrac{520}{729}f\left(-2k-1-\tfrac{N}{2}\right),\end{aligned}$$

and

$$\mathbf{U}^8\mathbf{f}(2k) = N^4\tfrac{4528}{6561}f(2k) - N^4\tfrac{1568}{729}f\left(2k+\tfrac{N}{2}\right), \quad (12)$$

$$\begin{aligned}\mathbf{U}^8\mathbf{f}(2k+1) &= N^4\tfrac{25696}{6561}f(2k+1)\\ &\quad - N^4\tfrac{784}{729}f\left(2k+1+\tfrac{N}{2}\right). \quad (13)\end{aligned}$$

From (3) and (12), it can be shown that the even components of the vector $[\mathbf{U}^8 - (r_1^4+r_2^4)\mathbf{U}^4 + r_1^4r_2^4\mathbf{I}]\mathbf{f}$

satisfy the equation $[\mathbf{U}^8-(r_1^4+r_2^4)\mathbf{U}^4+r_1^4r_2^4\mathbf{I}]\mathbf{f}(2k)=0$.

Likewise, it can be shown from (4) and (13) that the odd components $[\mathbf{U}^8-(r_1^4+r_2^4)\mathbf{U}^4+r_1^4r_2^4\mathbf{I}]\mathbf{f}(2k+1)=0$ for $k=0,1,\dots,\frac{1}{2}N-1$. Hence we get the minimal polynomial

$$Q(\lambda)=(\lambda^4-r_1^4)(\lambda^4-r_2^4),$$

from which the eigenvalues can exactly be obtained; thus establishing an equality in (11).

MULTIPLICITIES OF THE EIGENVALUES

Let the eigenvalue $\lambda_{pl}=r_p\,\mathrm{e}^{(\pi/2)il}$ and let m_{pl} denote the algebraic multiplicity of λ_{pl}, where $p=1,2$ and $l=0,1,2,3$. To obtain the multiplicities of the eigenvalues, we solve the linear system

$$\sum_{p=1}^{2}\sum_{l=0}^{3}m_{pl}=N, \tag{14}$$

$$\sum_{p=1}^{2}\sum_{l=0}^{3}m_{pl}\lambda_{pl}^n=\mathrm{tr}(\mathbf{U}^n), \tag{15}$$

where $n=1,2,\dots,7$. The system (14) and (15) can be reduced to (2×2) subsystems that can be solved using Cramer's rule. In order to evaluate the traces we require the following equations

$$\mathbf{U}\mathbf{f}(2k)=\frac{2}{3}\sum_{p=0}^{N/2-1}\omega^{4kp}f(2p)+\frac{4}{3}\sum_{p=0}^{N/2-1}\omega^{2k(2p+1)}f(2p+1), \tag{16}$$

$$\mathbf{U}\mathbf{f}(2k+1)=\frac{2}{3}\sum_{p=0}^{N/2-1}\omega^{(2k+1)2p}f(2p)+\frac{4}{3}\sum_{p=0}^{N/2-1}\omega^{(2k+1)(2p+1)}f(2p+1). \tag{17}$$

Applying the duality property, it can be shown that

$$\mathbf{U}^3\mathbf{f}(2k)=\frac{4}{27}N\sum_{p=0}^{N/2-1}\omega^{-4kp}f(2p)+\frac{40}{27}N\sum_{p=0}^{N/2-1}\omega^{-2k(2p+1)}f(2p+1),$$

Table 1 Traces.

n	$\mathrm{tr}(\mathbf{U}^n)$	n	$\mathrm{tr}(\mathbf{U}^n)$
1	$-\frac{2}{3}\sqrt{\frac{1}{2}N}\,\mathrm{e}^{(N-2)i\pi/8}$	2	$2N$
3	$-\frac{52}{27}N\sqrt{\frac{1}{2}N}\,\mathrm{e}^{-(N-2)i(\pi/8)}$	4	$\frac{98}{81}N^3$
5	$-\frac{808}{243}N^2\sqrt{\frac{1}{2}N}\,\mathrm{e}^{(N-2)i\pi/8}$	6	$\frac{88}{27}N^3$
7	$-\frac{11216}{2187}N^3\sqrt{\frac{1}{2}N}\,\mathrm{e}^{-(N-2)i\pi/8}$		

$$\mathbf{U}^3\mathbf{f}(2k+1)=\frac{20}{27}N\sum_{p=0}^{N/2-1}\omega^{-(2k+1)2p}f(2p)+\frac{56}{27}N\sum_{p=0}^{N/2-1}\omega^{-(2k+1)(2p+1)}f(2p+1).$$

Likewise, we can obtain expressions for the even and odd indexed components of $\mathbf{U}^5\mathbf{f}$ and $\mathbf{U}^7\mathbf{f}$. By writing (16) and (17) in matrix-vector form, we deduce that

$$\mathrm{tr}(\mathbf{U})=\frac{2}{3}\sum_{k=0}^{N/2-1}\omega^{(2k)^2}+\frac{4}{3}\sum_{k=0}^{N/2-1}\omega^{(2k+1)^2}. \tag{18}$$

The first sum in (18) can be separated as follows

$$\sum_{k=0}^{N/2-1}\omega^{4k^2}=1+\sum_{k=1}^{N/4-1/2}\omega^{4k^2}-\sum_{k=1}^{N/4-1/2}\omega^{(2k-1)^2}. \tag{19}$$

In a similar manner, we separate

$$\sum_{k=0}^{N/2-1}\omega^{(2k+1)^2}=-1-\sum_{k=1}^{N/4-1/2}\omega^{4k^2}+\sum_{k=1}^{N/4-1/2}\omega^{(2k-1)^2}. \tag{20}$$

Comparing (19) and (20), we note that $\sum_{k=0}^{N/2-1}\omega^{4k^2}=-\sum_{k=0}^{N/2-1}\omega^{(2k+1)^2}$. Hence, from (18), $\mathrm{tr}(\mathbf{U})=\frac{2}{3}\sum_{k=0}^{N/2-1}\omega^{(2k+1)^2}$. From the quadratic reciprocity law of Gauss sums[8], we have $\sum_{k=0}^{N/2-1}\omega^{(2k+1)^2}=-\sqrt{\frac{1}{2}N}\,\mathrm{e}^{(N-2)i\pi/8}$ which is used to evaluate the traces of odd multiples of $\mathbf{U}$. the traces of the matrices $\mathbf{U}^n$, $n=1,2,\dots,7$, are summarized in Table 1. The multiplicities of the eigenvalues are summarized in Table 2 and Table 3 for $N=4m+2$ where the integer $m\geqslant 2$.

EIGENVECTORS

An eigenvector $\mathbf{v}_{pl}$ corresponding to λ_{pl}, where $p=1,2$ and $l=0,1,2,3$ can be generated in the following manner using integral powers of $\mathbf{U}$; namely, $\mathbf{v}_{pl}=$

Table 2 Multiplicities corresponding to $\lambda_{1l}, l = 0, 1, 2, 3$.

Eigenvalue	Multiplicity
$\frac{1}{3}\sqrt{N}\sqrt{9+\sqrt{17}}$	$\frac{1}{2}[m+1-\cos(\frac{1}{2}m\pi)]$
$-\frac{1}{3}\sqrt{N}\sqrt{9+\sqrt{17}}$	$\frac{1}{2}[m+1+\cos(\frac{1}{2}m\pi)]$
$i\frac{1}{3}\sqrt{N}\sqrt{9+\sqrt{17}}$	$\frac{1}{2}[m-\sin(\frac{1}{2}m\pi)]$
$-i\frac{1}{3}\sqrt{N}\sqrt{9+\sqrt{17}}$	$\frac{1}{2}[m+\sin(\frac{1}{2}m\pi)]$

Table 3 Multiplicities corresponding to $\lambda_{2l}, l = 0, 1, 2, 3$.

Eigenvalue	Multiplicity
$\frac{1}{3}\sqrt{N}\sqrt{9-\sqrt{17}}$	$\frac{1}{2}[m+1+\cos(\frac{1}{2}m\pi)]$
$-\frac{1}{3}\sqrt{N}\sqrt{9-\sqrt{17}}$	$\frac{1}{2}[m+1-\cos(\frac{1}{2}m\pi)]$
$i\frac{1}{3}\sqrt{N}\sqrt{9-\sqrt{17}}$	$\frac{1}{2}[m+\sin(\frac{1}{2}m\pi)]$
$-i\frac{1}{3}\sqrt{N}\sqrt{9-\sqrt{17}}$	$\frac{1}{2}[m-\sin(\frac{1}{2}m\pi)]$

$\alpha_p \sum_{k=4}^{7} \lambda_{pl}^{-k} \mathbf{U}^k \mathbf{f} - \sum_{k=0}^{3} \lambda_{pl}^{-k} \mathbf{U}^k \mathbf{f}$ where $\alpha_1 = r_1^4/r_2^4$, $\alpha_2 = \alpha_1^{-1}$ and $\mathbf{f} \in \mathbb{C}^N$ is any non-zero vector, where $\mathbb{C}^N$ is the space of complex N-tuples. Since the minimal polynomial $Q(\mathbf{U})$ has linear elementary divisors, the matrix $\mathbf{U}$ is diagonalizable[4]; hence there exists a linearly independent set of eigenvectors of $\mathbf{U}$ that span $\mathbb{C}^N$.

The matrix $\mathbf{U}$ is not normal since $\mathbf{U}^*\mathbf{U} = \frac{1}{9}4N\,\mathrm{diag}[1,4,1,4,1\cdots,4]$ and

$$\mathbf{U}\mathbf{U}^* = \frac{2N}{9}\begin{pmatrix} 5\mathbf{I}_{N/2} & -3\mathbf{I}_{N/2} \\ -3\mathbf{I}_{N/2} & 5\mathbf{I}_{N/2} \end{pmatrix},$$

where $\mathbf{I}_{N/2}$ is the $(\frac{1}{2}N \times \frac{1}{2}N)$ identity matrix, hence there does not exist an orthogonal set of eigenvectors that span $\mathbb{C}^N$. In contrast, the DFT matrix possesses an orthogonal set of eigenvectors since it is unitary. Much research has focused on generating such an orthogonal set by using matrices that commute with the DFT matrix[9] and more recently a method based on complete generalized Legendre sequence has been proposed[10].

Theorem 1 *If* $\mathbf{f} \in \mathbb{C}^N$ *is an eigenvector of* $\mathbf{U}$*, then* $\mathbf{f}$ *is either even or odd.*

Proof: Assume that $\mathbf{f}$ is an eigenvector of $\mathbf{U}$ corresponding to some eigenvalue λ. From the duality property[7], for the even indexed components $\mathbf{U}^2 f(2k)$ we get

$$\tfrac{2}{3}Nf(-2k) - \tfrac{4}{9}Nf\left(-2k - \tfrac{N}{2}\right) = \lambda^2 f(2k) \quad (21)$$

and $\mathbf{U}^4 f(2k)$ gives

$$N^2\left[\tfrac{44}{81}f(2k) - \tfrac{8}{9}f\left(2k + \tfrac{N}{2}\right)\right] = \lambda^4 f(2k), \quad (22)$$

where $k = 0, 1, \dots, \frac{1}{2}N - 1$. Replacing k by $-k$ in (22) and using the periodicity N of $\mathbf{f}$ we obtain

$$N^2\left[\tfrac{44}{81}f(-2k) - \tfrac{8}{9}f\left(-2k - \tfrac{N}{2}\right)\right] = \lambda^4 f(-2k).$$

Solving for $f(-2k - \frac{1}{2}N)$ yields

$$f\left(-2k - \frac{N}{2}\right) = \frac{9}{8}\left[-\frac{\lambda^4}{N^2} + \frac{44}{81}\right] f(-2k) \quad (23)$$

and substituting (23) into (21) we get

$$f(-2k)\left[\frac{\lambda^4}{2N^2} + \frac{32}{81}\right] = \frac{\lambda^2}{N} f(2k). \quad (24)$$

Similarly it can be shown that the odd indexed components satisfy

$$f(2k+1)\left[\frac{\lambda^4}{2N^2} + \frac{32}{81}\right] f(-2k-1) = \frac{\lambda^2}{N^2}. \quad (25)$$

Substituting $\lambda = \lambda_{pl} = r_p\, \mathrm{e}^{i(\pi/2)l}$ into (24) and (25), we get

$$f(-2k)\left[\frac{r_p^4}{2N^2} + \frac{32}{81}\right] = \frac{r_p^2}{N}(-1)^l f(2k), \quad (26)$$

$$f(-2k-1)\left[\frac{r_p^4}{2N^2} + \frac{32}{81}\right] = \frac{r_p^2}{N^2}(-1)^l f(2k+1). \quad (27)$$

Consider the quartic equation $x^4/2N^2 + \frac{32}{81} = x^2/N$ in the variable x, which has only two positive solutions, namely r_1 and r_2, which leads us to conclude from (26) and (27) that

$$f(-2k) = (-1)^l f(2k),$$
$$f(-2k-1) = (-1)^l f(2k+1).$$

Hence we conclude that the eigenvectors are even for $l = 0, 2$ and odd for $l = 1, 3$. □

We further conclude from Theorem 1 that eigenvectors corresponding to real eigenvalues are even while those corresponding to purely imaginary eigenvalues are odd.

Theorem 2 *If* $\mathbf{f} \in \mathbb{C}^N$ *is an even eigenvector corresponding to* r_1*, then*

$$\mathbf{f} = (\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{y}$$

for some non-zero even vector $\mathbf{y} \in \mathbb{C}^N$.

Proof: Choose $\mathbf{y} = (1/2r_1(r_1^2 - r_2^2))\mathbf{f}$, then it is easily verified that

$$(\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\left[\frac{1}{2r_1(r_1^2 - r_2^2)}\mathbf{f}\right] = \mathbf{f}.$$

□

Theorem 3 *If* $\mathbf{f} \in \mathbb{C}^N$ *is an odd eigenvector corresponding to* $r_1 i$*, then*

$$\mathbf{f} = (\mathbf{U} + r_1 i\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{y}$$

for some non-zero odd vector $\mathbf{y} \in \mathbb{C}^N$.

Proof: Choose $\mathbf{y} = (1/2r_1 i(r_2^2 - r_1^2))\mathbf{f}$, then it is easily verified that

$$(\mathbf{U} + r_1 i\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I}) \left[\frac{1}{2r_1 i(r_2^2 - r_1^2)}\mathbf{f}\right] = \mathbf{f}.$$

□

Similar results can be proved for the eigenvectors corresponding to the remaining eigenvalues.

GENERATING LINEARLY INDEPENDENT EIGENVECTORS

Theorem 2 and Theorem 3 provide a means for numerically generating both even and odd eigenvectors. Define a sequence of even vectors

$$\mathbf{e}_{k+1} = \left[0, \dots, \underset{(k)^{\text{th}}}{1}, 0, \dots, \underset{(N-k)^{\text{th}}}{1}, 0, \dots\right]^{\mathrm{T}},$$

where $\mathbf{e}_{k+1} \in \mathbb{C}^N$ for $k = 0, 1, \dots, \frac{1}{2}N$, and a sequence of odd vectors

$$\mathbf{o}_{k+1} = \left[0, \dots, \underset{(k+2)^{\text{th}}}{1}, 0, \dots, \underset{(N-k-1)^{\text{th}}}{-1}, 0, \dots\right]^{\mathrm{T}},$$

where $\mathbf{o}_{k+1} \in \mathbb{C}^N$ for $k = 0, 1, \dots, \frac{1}{2}N - 2$. The sequence of vectors

$$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{N/2+1}, \mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_{N/2-1}\}$$

forms an orthogonal basis for $\mathbb{C}^N$ consisting of $\frac{1}{2}N+1$ even vectors and $\frac{1}{2}N - 1$ odd vectors[5].

We claim that the set

$$\{(\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}_k\}_{k=1}^{m_{10}-1;\nu_{10}},$$

where $\nu_{10} = 2m_{10} - 1$, of eigenvectors corresponding to r_1 is linearly independent. We assume without loss of generality that the dimension m_{10} of the corresponding eigenspace is even. Forming the linear combination

$$\sum_{k=1}^{m_{10}-1;\nu_{10}} \alpha_k(\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}_k = \mathbf{0},$$

defining $\mathbf{z} = \sum_{k=1}^{m_{10}-1;\nu_{10}} \alpha_k(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}_k$ and simplifying using the duality property we get

$$\mathbf{z} = [a\alpha_1, b\alpha_2, \dots, a\alpha_{m_{10}-1}, \dots, 0, \dots, a\alpha_{\nu_{10}}, \dots, \\ 0, \dots, a\alpha_{\nu_{10}}, \dots, 0, \dots, a\alpha_{m_{10}-1}, \dots, b\alpha_2]^{\mathrm{T}}$$

Table 4 An eigenbasis for $\mathbb{C}^N$.

Eigenvalue	**Eigenvectors**
r_1	$\{(\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}_k\}_{k=1}^{m_{10}-1;\nu_{10}}$
$-r_1$	$\{(\mathbf{U} - r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}_k\}_{k=1}^{m_{12}-1;\nu_{12}}$
ir_1	$\{(\mathbf{U} + ir_1\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{o}_k\}_{k=1}^{m_{11}-1;\nu_{11}}$
$-ir_1$	$\{(\mathbf{U} - ir_1\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{o}_k\}_{k=1}^{m_{13}-1;\nu_{13}}$
r_2	$\{(\mathbf{U} + r_2\mathbf{I})(\mathbf{U}^2 - r_1^2\mathbf{I})\mathbf{e}_k\}_{k=1}^{m_{20}-1;\nu_{20}}$
$-r_2$	$\{(\mathbf{U} - r_2\mathbf{I})(\mathbf{U}^2 - r_1^2\mathbf{I})\mathbf{e}_k\}_{k=1}^{m_{22}-1;\nu_{22}}$
ir_2	$\{(\mathbf{U} + ir_2\mathbf{I})(\mathbf{U}^2 + r_1^2\mathbf{I})\mathbf{o}_k\}_{k=1}^{m_{21}-1;\nu_{21}}$
$-ir_2$	$\{(\mathbf{U} - ir_2\mathbf{I})(\mathbf{U}^2 + r_1^2\mathbf{I})\mathbf{o}_k\}_{k=1}^{m_{23}-1;\nu_{23}}$

where $a = \frac{2}{3}N - r_2^2$ and $b = \frac{4}{3}N - r_2^2$. The matrix-vector system $(\mathbf{U} + r_1\mathbf{I})\mathbf{z} = \mathbf{0}$ can be simplified to yield the $(m_{10} \times m_{10})$ matrix-vector system $\mathbf{Cb} = \mathbf{0}$, where $\mathbf{C} = (\mathbf{C_1}|\mathbf{C_2})$, $\mathbf{C_1}$ is given by its elements $(\mathbf{C}_1)_{ij} = \cos jt_i$ for $i = 0, 1, \dots, m_{10} - 1$, $j = 0, 1, \dots, m_{10} - 2$, where $t_k = 4k\pi/N$, $k = 0, 1, \dots, m_{10} - 1$ are distinct points in the interval $[0, \pi]$, $\mathbf{C_2} = [+1, -1, +1, \dots, +1, -1]^{\mathrm{T}}$, and $\mathbf{b} = [a\alpha_1, -4b\alpha_2, 2a\alpha_3, \dots, -4b\alpha_{m_{10}-1}, 2a\alpha_{\nu_{10}}]^{\mathrm{T}}$. Using the fact that $\{1, \cos t, \cos 2t, \dots, \cos(m_{10} - 2)t\}$ is a Chebyshev set on $[0, \pi]$, noting the fact that the columns of $\mathbf{C}_1$ consists of this Chebyshev set and that the last column of $\mathbf{C}$ is non-zero and alternates in sign, we deduce that the matrix $\mathbf{C}$ is non-singular[6], and since a and b are non-zero, it follows that $\alpha_k = 0$ for $k = 1, 2, \dots, m_{10} - 1; \nu_{10}$, thus proving the linear independence of the eigenvectors. Hence we have succeeded in generating a basis for the eigenspace associated with r_1. Similar results hold true for the other eigenspaces and are summarized in Table 4, where $\nu_{pl} = 2m_{pl} - 1$, $p = 1, 2; l = 0, 1, 2, 3$ and $N = 4m + 2$.

Theorem 4 *Let* $\mathbf{f} = (\mathbf{U} + r_1\mathbf{I})(\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}$*, (where* $\mathbf{e} \in \mathbb{R}^N$ *is even) be an eigenvector corresponding to* r_1*, then* $\mathbf{f} \in \mathbb{R}^N$.

Proof: Let $\mathbf{z} = (\mathbf{U}^2 - r_2^2\mathbf{I})\mathbf{e}$, then $\mathbf{z}$ is real and from the duality property[7], it follows that $\mathbf{z}$ is even. To show that $(\mathbf{U} + r_1\mathbf{I})\mathbf{z}$ is real, it suffices to show that $\mathbf{Uz}$ is real. Conjugating $\mathbf{Uz}$ results in

$$\overline{\mathbf{Uz}}(k) = \frac{2}{3}\sum_{j=0}^{N/2-1} \omega^{-k2j} z(-2j) \\ + \frac{4}{3}\sum_{j=0}^{N/2-1} \omega^{-k(2j+1)} z(-2j-1)$$

$$= \frac{2}{3} \sum_{j=-N/2+1}^{0} \omega^{k2j} z(2j) + \frac{4}{3} \sum_{j=-N/2}^{-1} \omega^{k(2j+1)} z(2j+1). \quad (28)$$

Since the argument of the summations in (28) is periodic of period $\frac{1}{2}N$, it follows from the periodicity of the arguments of the summations[11] that $\overline{\mathbf{Uz}}(k)$ simplifies to

$$\frac{2}{3} \sum_{j=0}^{N/2-1} \omega^{k2j} z(2j) + \frac{4}{3} \sum_{j=0}^{N/2-1} \omega^{k(2j+1)} z(2j+1).$$

□

Theorem 5 *Let* $\mathbf{f} = (\mathbf{U} + ir_1\mathbf{I})(\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{o}$, *(where* $\mathbf{o} \in \mathbb{R}^N$ *is odd) be an eigenvector corresponding to* ir_1, *then* $\mathbf{f} \in \mathbb{C}^N$ *is purely imaginary.*

Proof: Let $\mathbf{z} = (\mathbf{U}^2 + r_2^2\mathbf{I})\mathbf{o}$, then $\mathbf{z}$ is real and from the duality property[7], it follows that $\mathbf{z}$ is odd. To show that $(\mathbf{U} + ir_1\mathbf{I})\mathbf{z}$ is purely imaginary, it suffices to show that $\mathbf{Uz}$ is purely imaginary.

$$\overline{\mathbf{Uz}}(k) = -\frac{2}{3} \sum_{j=0}^{N/2-1} \omega^{-k2j} z(-2j) - \frac{4}{3} \sum_{j=0}^{N/2-1} \omega^{-k(2j+1)} z(-2j-1).$$

Since $\mathbf{z}$ is odd and using the periodicity of the arguments of the summations[11], it follows that

$$\begin{aligned}\overline{\mathbf{Uz}}(k) &= -\frac{2}{3} \sum_{j=-N/2+1}^{0} \omega^{k2j} z(2j) - \frac{4}{3} \sum_{j=-N/2}^{-1} \omega^{k(2j+1)} z(2j+1) \\ &= -\frac{2}{3} \sum_{j=0}^{N/2-1} \omega^{k2j} z(2j) - \frac{4}{3} \sum_{j=0}^{N/2-1} \omega^{k(2j+1)} z(2j+1) \\ &= -\mathbf{Uz}(k).\end{aligned}$$

□

Hence by replacing $\mathbf{o}_k$ in Table 4 by $i\mathbf{o}_k$, we obtain a real eigenbasis for $\mathbb{C}^N$.

CONCLUSIONS

We notice from Table 2 and Table 3 that the discrete spectrum consists of four real and four purely imaginary eigenvalues. The normalized eigenvalues (normalized with respect to $\sqrt{N}$) of the DFT matrix lies on a unit circle in the complex plane. The normalized eigenvalues of the Simpson DFT matrix lie on the circumference of concentric circles with radii $\hat{r}_2 = \frac{1}{3}\sqrt{9 - \sqrt{17}} < 1$ and $\hat{r}_1 = \frac{1}{3}\sqrt{9 + \sqrt{17}} > 1$. The sum of the multiplicities of the eigenvalues corresponding to the real eigenvalues is $\frac{1}{2}N + 1$, while that corresponding to the purely imaginary eigenvalues is $\frac{1}{2}N - 1$. This is also true of the DFT matrix[5]. The sum of the multiplicities in Table 2 and Table 3 are the same, namely $\frac{1}{2}N$. The effect of the transformation on an eigenvector is to leave the direction unchanged, reverse it or rotate it clockwise or anticlockwise by $\frac{1}{2}\pi$. In addition there is a magnification factor of $\hat{r}_1$ or $\hat{r}_2$ which is absent in the DFT case. Incidentally Table 2 and Table 3 is also valid for $N = 6$. The minimal polynomial has two quartic factors that bear close resemblance to that of the minimal polynomial of the DFT matrix.

The space $\mathbb{C}^N$ is decomposed into the direct sum of eight subspaces, namely $\mathbb{C}^N = \mathcal{N}_{10} \oplus \mathcal{N}_{11} \oplus \cdots \oplus \mathcal{N}_{22}$, where $\mathcal{N}_{pl}$ is the nullspace of $(\mathbf{U} - \lambda_{pl}\mathbf{I})$, $p = 1, 2$ and $l = 0, 1, 2, 3$. Each subspace may be orthogonalized by the Gram-Schmidt process; however it must be stressed that these subspaces are not mutually orthogonal.

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