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Inequalities on Hardy and higher-power weighted Bergman spaces of composition operators

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ABSTRACT: Bounded composition operators are usually induced by analytic self-maps of the open unit disk acting on the Hardy space H^2 and on the higher-power weighted Bergman spaces $L^2_{e_\alpha}$ where $e_\alpha = (\alpha+1)^2 - 1$. An inequality for the relationship between the norms of the corresponding composition operators defined on these spaces is considered.

KEYWORDS: norm inequalities, semidefinite matrices, Schur product theorem

INTRODUCTION

Let D be the open unit disk in the complex plane and let $\varphi : D \to D$ be an analytic self-map. If $\mathcal{H}$ is a Hilbert space of analytic functions $f : D \to \mathbb{C}$, the composition operator C_φ on $\mathcal{H}$ is defined by $C_\varphi(f) = f \circ \varphi$ for all $f \in \mathcal{H}$. While there are some Hilbert spaces (for example, the Dirichlet space) where the composition operators are unbounded, every analytic φ induces a bounded operator on all of the spaces considered in this paper. We show relationships between the operator norms of C_φ acting on different spaces with weights.

The Hilbert spaces of primary interest to us will be the Hardy space H^2 and the power weighted Bergman spaces $L^2_{e_\alpha}$ where $e_\alpha = (\alpha+1)^2 - 1$. The Hardy space consists of all analytic functions f on D such that

$$\|f\|^2_{H^2} = \frac{1}{2\pi} \sup_{0<r<1} \int_0^{2\pi} \left|f(r\,\mathrm{e}^{i\theta})\right|^2 d\theta < \infty,$$

with the inner product

$$\langle f, g\rangle_{H^2} = \frac{1}{2\pi} \lim_{r\to 1^-} \int_0^{2\pi} f(r\,\mathrm{e}^{i\theta})\overline{g(r\,\mathrm{e}^{i\theta})} d\theta < \infty.$$

The Hardy space can be described as a reproducing kernel Hilbert space, since for every point $\lambda \in D$ there is a unique function $K_\lambda \in H^2$ such that $\langle f, K_\lambda\rangle_{H^2} = f(\lambda)$ for all $f \in H^2$; in fact, $K_\lambda(z) = 1/(1-\overline{\lambda}z)$ (see Ref. 1).

For $\alpha > -1$, we define the power weighted Bergman space, denoted $L^2_{e_\alpha}$, to be the space of all analytic functions f on D such that

$$\|f\|^2_{L^2_{e_\alpha}} = \int_D |f(z)|^2 (\alpha+1)^2 (1-|z|^2)^{e_\alpha}\, \mathrm{d}A < \infty,$$

where $\mathrm{d}A$ is the normalized area measure on D.

We write $\langle\cdot,\cdot\rangle_{L^2_{e_\alpha}}$, for any α, to denote the inner product on $L^2_{e_\alpha}$ with the kernel function $k^{e_\alpha}_\lambda(z) = 1/(1-\overline{\lambda}z)^{e_\alpha+2}$. There is an obvious likeness between the reproducing kernels for H^2 and the analogous functions for $L^2_{e_\alpha}$. For the sake of efficiency, we write L^2_{-1} to denote the Hardy space H^2, with $k^{-1}_\lambda = K_\lambda$ and $\langle\cdot,\cdot\rangle_{L^2_{-1}} = \langle\cdot,\cdot\rangle_{H^2}$. We will state many of the results in these terms, with the understanding that the $\alpha = 0$ and $\alpha = -2$ power weighted Bergman spaces always signifies the Hardy space.

For any analytic $\varphi : D \to D$, we will write $\|C_\varphi\|_{\mathcal{H}}$ to denote the norm of C_φ acting on a Hilbert space $\mathcal{H}$. While, it is generally not easy to calculate the norm $\|C_\varphi\|_{L^2_{e_\alpha}}$ explicitly[2–5], it is in fact not difficult to estimate the norm of C_φ. In particular, it is well known that

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{e_\alpha+2} \leqslant \|C_\varphi\|^2_{L^2_{e_\alpha}} \leqslant \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|^2}\right)^{e_\alpha+2} \qquad (1)$$

for any $\alpha \geqslant -1$ (see Refs. 1, 6). In spite of (1), one might wonder whether there is some relationship between the quantities $\|C_\varphi\|_{L^2_{e_\alpha}}$ for different values of α.

For example, considering $\alpha = 0$, $\alpha = -1$ and $\alpha = -2$, one might ask whether it is always the case that $\|C_\varphi\|_{L^2} = \|C_\varphi\|^2_{H^2}$. While this equality does hold for some maps, it is not true in general[7]. Christopher Hammond and Linda J. Patton[8] proved that $\|C_\varphi\|_{L^2} \leqslant \|C_\varphi\|^2_{H^2}$ for all φ answering a question posed by Carswell and Hammond[7], and they derived a collection of inequalities relating to the norms of C_φ acting on different spaces.

In this paper we apply norm inequalities for composition operators[8] to give a verification of higher-power weighted Bergman spaces. Now we should mention a helpful fact relating to composition operators and reproducing kernel functions. Let C^*_φ denote the adjoint of C_φ on a particular space $L^2_{e_\alpha}$, and that $C^*_\varphi(K^{e_\alpha}_\lambda) = K^{e_\alpha}_{\varphi(\lambda)}$ for any $\lambda \in D$ (see Ref. 1). This observation will provide exactly the verification of the information we need to compare the action of C_φ on different spaces.

POSITIVE SEMIDEFINITE MATRICES

Let $\Lambda = \{\lambda_m\}^\infty_{m=1}$, a sequence of distinct points in D, be a set of uniqueness for the collection of analytic functions on D. In other words, the zero function is the only analytic function with $f(\lambda_m) = 0$ for all m. The span of the kernel functions $\{k^{e_\alpha}_{\lambda_m}\}^\infty_{m=1}$ is dense in every space $L^2_{e_\alpha}$, since any function orthogonal to every $k^{e_\alpha}_{\lambda_m}$ must be identically 0. Throughout this paper, we will assume that such a sequence Λ has been fixed.

Consider an analytic map $\varphi : D \to D$. For a positive constant ν, a natural number n, and a real number $\alpha \geqslant -1$, we define the $n \times n$ matrix $M(\nu, n, \alpha(\alpha+2)) = (m_{ij})_{n\times n}$ by

$$m_{ij} = \frac{\nu^2}{\left(1 - \overline{\lambda}_j\lambda_i\right)^{e_\alpha+2}} - \frac{1}{\left(1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i)\right)^{e_\alpha+2}}$$

where $e_\alpha = (\alpha+1)^2 - 1$. In particular, we put

$$M = \operatorname{diag}\left(\frac{\nu^2}{\left(1 - |\lambda|^2\right)^{e_\alpha+2}} - \frac{1}{\left(1 - |\varphi(\lambda_j)|^2\right)^{e_\alpha+2}}\right).$$

Recall that an $n \times n$ matrix A is called positive semi-definite if $\langle Ac, c\rangle \geqslant 0$ for all $c \in \mathbb{C}^n$, denoted $A \geqslant 0$ where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product. Any such matrix must necessarily be self-adjoint. For self-adjoint matrices A and B, we write $A \geqslant B$ if $A - B \geqslant 0$. The following proposition relates $\|C_\varphi\|_{L^2_{e_\alpha}}$ to the positive semi-definiteness of $M(\nu, n, \alpha(\alpha+2))$.

Proposition 1 *Let $\varphi : D \to D$ be an analytic self-map and n be a positive constant. Then, for any $\alpha \geqslant -1$, the matrix $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite for all natural numbers n if and only if $\|C_\varphi\|_{L^2_{e_\alpha}} \leqslant \nu$.*

Proof: Assume first that $\|C_\varphi\|_{L^2_{e_\alpha}} \leqslant \nu$, from which it follows that $\left\|C^*_\varphi\right\|_{L^2_{e_\alpha}} \leqslant \nu$. In other words, we have

$$\left\|C^*_\varphi(f)\right\|^2_{L^2_{e_\alpha}} \leqslant \nu^2 \|f\|^2_{L^2_{e_\alpha}}. \tag{2}$$

Let $f \in L^2_{e_\alpha}$ and $c_1, \dots, c_n \in \mathbb{C}$. We express $f = \sum^n_{j=1} c_j k^{e_\alpha}_{\lambda_j}$. If we substitute this function into (2), recalling that $C^*_\varphi(k^{e_\alpha}_\lambda) = k^{e_\alpha}_{\varphi(\lambda)}$, then we obtain

$$\left\|\sum^n_{j=1} c_j k^{e_\alpha}_{\varphi(\lambda_j)}\right\|^2_{L^2_{e_\alpha}} \leqslant \nu^2 \left\|\sum^n_{j=1} c_j k^{e_\alpha}_{\lambda_j}\right\|^2_{L^2_{e_\alpha}}$$

from which it follows that

$$\sum^n_{j=1} |c_j|^2 \left\|k^{e_\alpha}_{\varphi(\lambda_j)}\right\|^2_{L^2_{e_\alpha}} \leqslant \sum^n_{j=1} \nu^2 |c_j|^2 \left\|k^{e_\alpha}_{\lambda_j}\right\|^2_{L^2_{e_\alpha}}$$

and thus

$$\sum^n_{j=1} |c_j|^2 \left[\frac{\nu^2}{(1 - |\lambda_j|^2)^{e_\alpha+2}} - \frac{1}{(1 - |\varphi(\lambda_j)|^2)^{e_\alpha+2}}\right] \geqslant 0. \tag{3}$$

Inequality (3) precisely implies that $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite.

For the converse, assume that $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite for all natural numbers n. Hence (3) holds for all n, which in turn implies that

$$\left\|\sum^n_{j=1} c_j k^{e_\alpha}_{\varphi(\lambda_j)}\right\|^2_{L^2_{e_\alpha}} \leqslant \nu^2 \left\|\sum^n_{j=1} c_j k^{e_\alpha}_{\lambda_j}\right\|^2_{L^2_{e_\alpha}}. \tag{4}$$

For any n and any complex numbers $c_1, \dots, c_n$, let f be an arbitrary element of $L^2_{e_\alpha}$. Since Λ is a set of uniqueness, the span of $\{k^{e_\alpha}_{\lambda_n}\}^\infty_{n=1}$ is dense in $L^2_{e_\alpha}$. Hence there exists a sequence $\{f_m\}^\infty_{m=1}$ that converges to f in norm, where each f_m is a finite linear combination of these kernel functions. The inequality of (4) implies that $\left\|C^*_\varphi(f_m)\right\|^2_{L^2_{e_\alpha}} \leqslant \nu^2 \|f_m\|^2_{L^2_{e_\alpha}}$ for all m.

Letting $m \to \infty$, we see that $\left\|C^*_\varphi(f)\right\|^2_{L^2_{e_\alpha}} \leqslant \nu^2 \|f\|^2_{L^2_{e_\alpha}}$, from which it follows (upon taking the supremum over all $f \in L^2_{e_\alpha}$) that

$$\|C_\varphi\|_{L^2_{e_\alpha}} = \left\|C^*_\varphi\right\|_{L^2_{e_\alpha}} \leqslant \nu.$$

Hence Proposition 1 states that $\|C_\varphi\|_{L^2_{e_\alpha}} \leqslant \nu$ exactly when

$$k_\lambda(z) = \frac{\nu^2}{(1-\overline{\lambda}z)^{e_\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda)}\varphi(z))^{e_\alpha+2}}$$

is a positive semi definite kernel on the unit disk. □

Remark 1 If $f^r = \sum_{j=1}^n c_j k_{\lambda_j}^{(\alpha_r+1)^2-1}$ where $r = 1, 2, \dots, n$. Proposition 1 implies that $f_m^r \to f^r$ uniformly in the norm. We can deduce that

$$\left\| c^* \left(\sum_{r=1}^n f^r \right) \right\|^2_{L^2_{(\alpha_r+1)^2-1}} \leqslant \nu^2 \sum_{r=1}^n \|f^r\|^2_{L^2_{(\alpha_r+1)^2-1}}.$$

We need the following lemma which relating to positive semi-definite matrices.

Lemma 1 *Let $\lambda_1, \dots, \lambda_n$ be a finite collection of (not necessarily distinct) points in D. Any matrix of the form*

$$M = \left[\frac{1}{(1-\overline{\lambda_j}\lambda_i)^\rho} \right]_{i,j=1}^n,$$

for any real number $\rho \geqslant 1$, must be positive semi-definite, and so is a diagonal matrix

$$\operatorname{diag}\left(\frac{1}{(1-|\lambda_j|^2)^\rho} \right)_{j=1}^n.$$

Proof: Let $\alpha = \sqrt{\rho-1} - 1$ so that $\alpha \geqslant -1$. Taking $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, we see that

$$\begin{aligned} \langle Mc, c \rangle &= \sum_{i=1}^n \sum_{j=1}^n \frac{\overline{c}_i c_j}{(1-\overline{\lambda}_j\lambda_i)^{(\alpha+1)^2+1}} \\ &= \left\langle \sum_{j=1}^n c_j k_{\lambda_j}^{(\alpha+1)^2-1}, \sum_{i=1}^n c_i k_{\lambda_i}^{(\alpha+1)^2-1} \right\rangle_{L^2_{(\alpha+1)^2-1}} \\ &\geqslant 0, \end{aligned}$$

from which our assertion follows and

$$\langle Mc, c \rangle = \left\| \sum_{j=1}^n c_j k_{\lambda_j}^{(\alpha+1)^2-1} \right\|^2_{L^2_{(\alpha+1)^2-1}}.$$

As a consequence of Lemma 1, we see that any matrix of the form

$$\left[\frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^\rho} \right]_{i,j=1}^n,$$

where φ is a self-map of D, must also be positive semi-definite and so is

$$\left[\frac{1}{(1-|\varphi(\lambda_j)|^2)^\rho} \right]_{j=1}^n$$

as required. □

NORM INEQUALITIES

The proof of the major theorem relies heavily on the use of Schur products. Recall that, for any two $n \times n$ matrices $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$, the Schur (or Hadamard) product $A \circ B$ is defined by the following rule $A \circ B = [a_{ij}b_{ij}]_{i,j=1}^n$. That is, the Schur product is obtained by entrywise multiplication. A proof of the following result appears in Ref. 9.

Proposition 2 (Schur Product Theorem) *If A and B are $n \times n$ positive semi-definite matrices, then $A \circ B$ is also positive semi-definite.*

We are now in position to state the main result, a theorem that allows us to compare the norms of C_φ on certain weighted spaces.

Theorem 1 *Take $\beta \geqslant e_\alpha := (\alpha+1)^2 - 1 \geqslant -1$ and let φ be an analytic self-map of D. Then*

$$\|C_\varphi\|_{L^2_\beta} \leqslant \|C_\varphi\|^\gamma_{L^2_{e_\alpha}},$$

whenever the quantity $\gamma := (\beta+2)/(e_\alpha+2)$ is an integer.

Proof: Assume that $\gamma = (\beta+2)/(e_\alpha+2)$ is an integer. Fix a natural number n and let $i, j \in \{1, 2, \dots, n\}$. A difference of higher powers factorization shows that

$$\begin{aligned} &\frac{\|C_\varphi\|^{2\gamma}_{L^2_{e_\alpha}}}{(1-\overline{\lambda_j}\lambda_i)^{\beta+2}} - \frac{1}{\left(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i)\right)^{\beta+2}} \\ &= \left(\frac{\|C_\varphi\|^2_{L^2_{e_\alpha}}}{(1-\overline{\lambda_j}\lambda_i)^{e_\alpha+2}} - \frac{1}{\left(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i)\right)^{e_\alpha+2}} \right) \\ &\qquad \times \sum_{k=0}^{\gamma-1} \left(\frac{\|C_\varphi\|^{2k}_{L^2_{e_\alpha}}}{c_k d_k} \right) \end{aligned}$$

where $c_k = (1-\overline{\lambda_j}\lambda_i)^{k(e_\alpha+2)}$ and $d_k = (1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(e_\alpha+2)(\gamma-k-1)}$. Then

$$\begin{aligned} &\frac{\|C_\varphi\|^{2\gamma}_{L^2_{e_\alpha}}}{(1-|\lambda_j|^2)^{\beta+2}} - \frac{1}{(1-|\varphi(\lambda_j)|^2)^{\beta+2}} \\ &= \left(\frac{\|C_\varphi\|^2_{L^2_{e_\alpha}}}{(1-|\lambda_j|^2)^{e_\alpha+2}} - \frac{1}{(1-|\varphi(\lambda_j)|^2)^{e_\alpha+2}} \right) \\ &\qquad \times \sum_{k=0}^{\gamma-1} \left(\frac{\|C_\varphi\|^{2k}_{L^2_{e_\alpha}}}{a_k b_k} \right) \end{aligned}$$

where $a = (1 - |\lambda_j|^2)^{k(e_\alpha+2)}$ and $b = (1 - |\varphi(\lambda_j)|^2)^{(e_\alpha+2)(\gamma-k-1)}$.

Since the preceding equation holds for all i and j, we obtain the following matrix equation:

$$M\left(\|C_\varphi\|^\gamma_{L^2_{e_\alpha}}, n, \beta\right) = M\left(\|C_\varphi\|_{L^2_{e_\alpha}}, n, \alpha(\alpha+2)\right) \times \sum_{k=0}^{\gamma-1} \left(\frac{\|C_\varphi\|^{2k}_{L^2_{e_\alpha}}}{e_k f_k}\right)^n_{i,j=1} \quad (5)$$

where $e_k = (1 - \overline{\lambda_j}\lambda_i)^{k(e_\alpha+2)}$ and $f_k = (1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(e_\alpha+2)(\gamma-k-1)}$. This implies the matrix $M\left(\|C_\varphi\|_{L^2_{e_\alpha}}, n, \alpha(\alpha+2)\right)$ is positive semi-definite by Proposition 1.

Lemma 1, together with Proposition 2, dictates that every term in the matrix sum on the right-hand side of (5) is positive semi-definite, so the sum itself is positive semi-definite. Therefore Proposition 1 shows that $M\left(\|C_\varphi\|^\gamma_{L^2_{e_\alpha}}, n, \beta\right)$ must also be positive semi-definite.

Since this assertion holds for every natural number n, we obtain by Proposition 1 that $\|C_\varphi\|_{L^2_\beta} \leqslant \|C_\varphi\|^\gamma_{L^2_{e_\alpha}}$. □

Taking $\alpha = 0$, $\alpha = -1$ and $\alpha = -2$, we obtain[8] the following corollaries.

Corollary 1 *Let φ be an analytic self-map of D. Then*

$$\|C_\varphi\|_{L^2_\beta} \leqslant \|C_\varphi\|^{\beta+2}_{H^2},$$

whenever β is a non-negative integer. In particular, $\|C_\varphi\|_{L^2} \leqslant \|C_\varphi\|^2_{H^2}$.

Corollary 2 *Let φ be an analytic self-map of D. Then*

$$\|C_\varphi\|_{L^2_\beta} \leqslant \|C_\varphi\|^{(\beta+2)/2}_{L^2},$$

whenever β is a positive even integer.

Theorem 2 *Take $\beta \geqslant e_\alpha := (\alpha+1)^2 - 1 \geqslant -1$ and let φ be an analytic self-map of D. Suppose that $\gamma = (\beta+2)/(e_\alpha+2)$ is an integer. If C_φ is cosubnormal on $L^2_{e_\alpha}$, then it is also cosubnormal on L^2_β.*

Cowen[10] only stated this result for $\alpha = -1$, but an identical argument works for $\alpha > -1$. The proof makes use of Proposition 1 in a similar fashion to that of Theorem 1[8].

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