Inequalities on Hardy and higher-power weighted Bergman spaces of composition operators

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ABSTRACT: Bounded composition operators are usually induced by analytic self-maps of the open unit disk acting on the Hardy space \(H^2\) and on the higher-power weighted Bergman spaces \(L^{2}_{\alpha}\) where \(e_\alpha = (\alpha + 1)^2 - 1\). An inequality for the relationship between the norms of the corresponding composition operators defined on these spaces is considered.

KEYWORDS: norm inequalities, semidefinite matrices, Schur product theorem

INTRODUCTION

Let \(D\) be the open unit disk in the complex plane and let \(\varphi : D \rightarrow D\) be an analytic self-map. If \(\mathcal{H}\) is a Hilbert space of analytic functions \(f : D \rightarrow \mathbb{C}\), the composition operator \(C_\varphi\) on \(\mathcal{H}\) is defined by \(C_\varphi(f) = f \circ \varphi\) for all \(f \in \mathcal{H}\). While there are some Hilbert spaces (for example, the Dirichlet space) where the composition operators are unbounded, every analytic \(\varphi\) induces a bounded operator on all of the spaces considered in this paper. We show relationships between the operator norms of \(C_\varphi\) acting on different spaces with weights.

The Hilbert spaces of primary interest to us will be the Hardy space \(H^2\) and the power weighted Bergman spaces \(L^{2}_{\alpha}\) where \(e_\alpha = (\alpha + 1)^2 - 1\). The Hardy space consists of all analytic functions \(f\) on \(D\) such that

\[\|f\|_{H^2}^2 = \frac{1}{2\pi} \sup_{0 < r < 1} \int_{0}^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty,\]

with the inner product

\[
\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \lim_{r \to 1^-} \int_{0}^{2\pi} f(re^{i\theta})g(re^{i\theta}) \, d\theta < \infty.
\]

The Hardy space can be described as a reproducing kernel Hilbert space, since for every point \(\lambda \in D\) there is a unique function \(K_\lambda \in H^2\) such that \(\langle f, K_\lambda \rangle_{H^2} = f(\lambda)\) for all \(f \in H^2\); in fact, \(K_\lambda(z) = 1/(1-\overline{\lambda}z)\) (see Ref. 1).

For \(\alpha > -1\), we define the power weighted Bergman space, denoted \(L^{2}_{\alpha}\), to be the space of all analytic functions \(f\) on \(D\) such that

\[\|f\|_{L^{2}_{\alpha}}^2 = \int_{D} |f(z)|^2 (\alpha + 1)^2 (1 - |z|^2)^{e_\alpha} \, dA < \infty,\]

where \(dA\) is the normalized area measure on \(D\).

We write \(\langle \cdot, \cdot \rangle_{L^{2}_{\alpha}}\), for \(\alpha\), to denote the inner product on \(L^{2}_{\alpha}\) with the kernel function \(k^{e_\alpha}_\lambda(z) = 1/(1-\overline{\lambda}z)^{e_\alpha+2}\). There is an obvious likeness between the reproducing kernels for \(H^2\) and the analogous functions for \(L^{2}_{\alpha}\). For the sake of efficiency, we write \(L^{2}_{-1}\) to denote the Hardy space \(H^2\), with \(k^{-1}_\lambda = K_\lambda\) and \(\langle \cdot, \cdot \rangle_{L^{2}_{-1}} = \langle \cdot, \cdot \rangle_{H^2}\). We will state many of the results in these terms, with the understanding that the \(\alpha = 0\) and \(\alpha = -2\) power weighted Bergman spaces always signifies the Hardy space.

For any analytic \(\varphi : D \rightarrow D\), we will write \(\|C_\varphi\|_{\mathcal{H}}\) to denote the norm of \(C_\varphi\) acting on a Hilbert space \(\mathcal{H}\). While, it is generally not easy to calculate the norm \(\|C_\varphi\|_{L^{2}_{\alpha}}\) explicitly\textsuperscript{2-5}, it is in fact not difficult to estimate the norm of \(C_\varphi\). In particular, it is well known that

\[
\left(\frac{1}{1 - |\varphi(0)|^2}\right)^{e_\alpha+2} \leq \|C_\varphi\|_{L^{2}_{\alpha}}^2 \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|^2}\right)^{e_\alpha+2} \tag{1}
\]

for any \(\alpha \geq -1\) (see Refs. 1, 6). In spite of (1), one might wonder whether there is some relationship between the quantities \(\|C_\varphi\|_{L^{2}_{\alpha}}\) for different values of \(\alpha\).
For example, considering $\alpha = 0$, $\alpha = -1$ and $\alpha = -2$, one might ask whether it is always the case that $\|C_\varphi\|_{L^2} = \|C_\varphi^*\|_{H^2}$. While this equality does hold for some maps, it is not true in general\(^7\). Christopher Hammond and Linda J. Patton\(^8\) proved that $\|C_\varphi\|_{L^2} \leq \|C_\varphi^*\|_{H^2}$ for all $\varphi$ answering a question posed by Carswell and Hammond\(^7\), and they derived a collection of inequalities relating to the norms of $C_\varphi$ acting on different spaces.

In this paper we apply norm inequalities for composition operators\(^8\) to give a verification of higher-power weighted Bergman spaces. Now we should mention a helpful fact relating to composition operators and reproducing kernel functions. Let $C_\varphi$ denote the adjoint of $C_\varphi$ on a particular space $L^2_\nu$, and that $C_\varphi^*(K_\lambda^\alpha) = K_\lambda^\alpha$ for any $\lambda \in D$ (see Ref. 1). This observation will provide exactly the verification of the information we need to compare the action of $C_\varphi$ on different spaces.

**POSITIVE SEMIDEFINITE MATRICES**

Let $\Lambda = \{\lambda_m\}_{m=1}^{\infty}$, a sequence of distinct points in $D$, be a set of uniqueness for the collection of analytic functions on $D$. In other words, the zero function is the only analytic function with $f(\lambda_m) = 0$ for all $m$. The span of the kernel functions $\{K_\lambda^\alpha\}_{\lambda \in \Lambda}$ is dense in every space $L^2_\nu$, since any function orthogonal to every $K_\lambda^\alpha$ must be identically 0. Throughout this paper, we will assume that such a sequence $\Lambda$ has been fixed.

Consider an analytic map $\varphi : D \to D$. For a positive constant $\nu$, a natural number $n$, and a real number $\alpha \geq -1$, we define the $n \times n$ matrix $M(\nu, n, \alpha(\alpha+2)) = (m_{ij})_{n \times n}$ by

$$m_{ij} = \frac{\nu^2}{(1 - \lambda_j \lambda_i)^{\alpha+2}} - \frac{1}{(1 - \varphi(\lambda_j) \varphi(\lambda_i))^{\alpha+2}},$$

where $e_\alpha = (\alpha + 1)^2 - 1$. In particular, we put

$$M = \text{diag} \left( \frac{\nu^2}{(1 - |\lambda|^2)^{\alpha+2}} - \frac{1}{(1 - |\varphi(\lambda)|^2)^{\alpha+2}} \right).$$

Recall that an $n \times n$ matrix $A$ is called positive semi-definite if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$, denoted $A \succeq 0$ where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. Any such matrix must necessarily be self-adjoint. For self-adjoint matrices $A$ and $B$, we write $A \succeq B$ if $A - B \succeq 0$. The following proposition relates $\|C_\varphi\|_{L^2}$ to the positive semi-definiteness of $M(\nu, n, \alpha(\alpha+2))$.

**Proposition 1** Let $\varphi : D \to D$ be an analytic self-map and $n$ be a positive constant. Then, for any $\alpha \geq -1$, the matrix $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite for all natural numbers $n$ if and only if $\|C_\varphi\|_{L^2_\nu} \leq \nu$.

**Proof:** Assume first that $\|C_\varphi\|_{L^2_\nu} \leq \nu$, from which it follows that $\|C_\varphi^*\|_{L^2_\nu} \leq \nu$. In other words, we have

$$\|C_\varphi^*(f)\|_{L^2_\nu}^2 \leq \nu^2 \|f\|_{L^2_\nu}^2. \quad (2)$$

Let $f \in L^2_\nu$ and $c_1, \ldots, c_n \in \mathbb{C}$. We express $f = \sum_{j=1}^n c_j K^\alpha_{\lambda_j}$. If we substitute this function into (2), recalling that $C_\varphi^*(k^\alpha_{\lambda}) = k^\alpha_{\varphi(\lambda)}$, then we obtain

$$\left\| \sum_{j=1}^n c_j K^\alpha_{\varphi(\lambda_j)} \right\|_{L^2_\nu}^2 \leq \nu^2 \left\| \sum_{j=1}^n c_j k^\alpha_{\lambda_j} \right\|_{L^2_\nu}^2$$

from which it follows that

$$\sum_{j=1}^n |c_j|^2 \left\| k^\alpha_{\varphi(\lambda_j)} \right\|_{L^2_\nu}^2 \leq \sum_{j=1}^n \nu^2 |c_j|^2 \left\| k^\alpha_{\lambda_j} \right\|_{L^2_\nu}^2$$

and thus

$$\sum_{j=1}^n |c_j|^2 \left( \frac{\nu^2}{(1 - |\lambda_j|^2)^{\alpha+2}} - \frac{1}{(1 - |\varphi(\lambda_j)|^2)^{\alpha+2}} \right) \geq 0. \quad (3)$$

Inequality (3) precisely implies that $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite.

For the converse, assume that $M(\nu, n, \alpha(\alpha+2))$ is positive semi-definite for all natural numbers $n$. Hence (3) holds for all $n$, which in turn implies that

$$\left\| \sum_{j=1}^n c_j K^\alpha_{\varphi(\lambda_j)} \right\|_{L^2_\nu}^2 \leq \nu^2 \left\| \sum_{j=1}^n c_j k^\alpha_{\lambda_j} \right\|_{L^2_\nu}^2. \quad (4)$$

For any $n$ and any complex numbers $c_1, \ldots, c_n$, let $f$ be an arbitrary element of $L^2_\nu$. Since $\Lambda$ is a set of uniqueness, the span of $\{k^\alpha_{\lambda_m}\}_{m=1}^{\infty}$ is dense in $L^2_\nu$. Hence there exists a sequence $\{f_m\}_{m=1}^{\infty}$ that converges to $f$ in norm, where each $f_m$ is a finite linear combination of these kernel functions. The inequality of (4) implies that $\|C_\varphi^*(f_m)\|_{L^2_\nu}^2 \leq \nu^2 \|f_m\|_{L^2_\nu}^2$ for all $m$.

Letting $m \to \infty$, we see that $\|C_\varphi^*(f)\|_{L^2_\nu}^2 \leq \nu^2 \|f\|_{L^2_\nu}^2$, from which it follows (upon taking the supremum over all $f \in L^2_\nu$) that

$$\|C_\varphi\|_{L^2_\nu} = \|C_\varphi^*\|_{L^2_\nu} \leq \nu.$$
Hence Proposition 1 states that \( \|C_\varphi\|_{L^2_\alpha} \leq \nu \) exactly when
\[
k_\lambda(z) = \frac{\nu^2}{(1 - \lambda z)^{\alpha+2}} - \frac{1}{(1 - |\varphi(\lambda)|^2)|\varphi(\lambda)|^{\alpha+2}}
\]
is a positive semi definite kernel on the unit disk. \( \Box\)

**Remark 1** If \( f^r = \sum_{j=1}^n c_j k_{\lambda_j}^{\alpha}z^{-1} \) where \( r = 1, 2, \ldots, n \). Proposition 1 implies that \( f^r \rightarrow f^r \)
uniformly in the norm. We can deduce that
\[
\left\| \sum_{j=1}^n f^r_j \right\|^2_{L^2_{\alpha+2}} \leq \nu^2 \sum_{j=1}^n \|f^r\|_{L^2_{\alpha+2}}^2.
\]

We need the following lemma which relating to positive semi-definite matrices.

**Lemma 1** Let \( \lambda_1, \ldots, \lambda_n \) be a finite collection of (not necessarily distinct) points in \( D \). Any matrix of the form
\[
M = \left[ \frac{1}{(1 - \lambda_j^\alpha)^\rho} \right]_{i,j=1}^n
\]
for any real number \( \rho \geq 1 \), must be positive semi-definite, and so is a diagonal matrix
\[
diag \left( \frac{1}{(1 - |\lambda_j|^\alpha)^\rho} \right)_{j=1}^n.
\]

**Proof:** Let \( \alpha = \sqrt{p-1} - 1 \) so that \( \alpha \geq -1 \). Taking \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \), we see that
\[
\langle Mc, c \rangle = \sum_{i=1}^n \sum_{j=1}^n \frac{c_i c_j}{(1 - \lambda_j \lambda_i)^{\alpha+2}} = \sum_{j=1}^n c_j k_{\lambda_j}^{\alpha}z^{-1} \sum_{i=1}^n c_i k_{\lambda_i}^{\alpha}z^{-1}
\]
\[
\geq 0,
\]
from which our assertion follows and
\[
\langle Mc, c \rangle = \left\| \sum_{j=1}^n c_j k_{\lambda_j}^{\alpha}z^{-1} \right\|^2_{L^2_{\alpha+2}}.
\]

As a consequence of Lemma 1, we see that any matrix of the form
\[
\left[ \frac{1}{(1 - |\varphi(\lambda_j)|^\alpha)^\rho} \right]_{i,j=1}^n,
\]
where \( \varphi \) is a self-map of \( D \), must also be positive semi-definite and so is
\[
\left[ \frac{1}{(1 - |\varphi(\lambda_j)|^\alpha)^\rho} \right]_{j=1}^n
\]
as required. \( \Box\)

**NORM INEQUALITIES**

The proof of the major theorem relies heavily on the use of Schur products. Recall that, for any two \( n \times n \) matrices \( A = [a_{ij}]_{i,j=1}^n \) and \( B = [b_{ij}]_{i,j=1}^n \), the Schur (or Hadamard) product \( A \circ B \) is defined by the following rule \( A \circ B = [a_{ij}b_{ij}]_{i,j=1}^n \). That is, the Schur product is obtained by entrywise multiplication. A proof of the following result appears in Ref. 9.

**Proposition 2** (Schur Product Theorem) If \( A \) and \( B \) are \( n \times n \) positive semi-definite matrices, then \( A \circ B \) is also positive semi-definite.

We are now in position to state the main result, a theorem that allows us to compare the norms of \( C_\varphi \) on certain weighted spaces.

**Theorem 1** Take \( \beta \geq \gamma := (\beta + 2)/\alpha \) and let \( \varphi \) be an analytic self-map of \( D \). Then
\[
\|C_\varphi\|_{L^2_\beta} \leq \|C_\varphi\|_{L^2_\alpha}^{\gamma},
\]
whenever the quantity \( \gamma := (\beta + 2)/\alpha \) is an integer.

**Proof:** Assume that \( \gamma = (\beta + 2)/\alpha \) is an integer. Fix a natural number \( n \) and let \( i, j \in \{1, 2, \ldots, n\} \). A difference of higher powers factorization shows that
\[
\|C_\varphi\|_{L^2_\alpha}^{2\gamma} \leq \|C_\varphi\|_{L^2_\alpha} \leq \|C_\varphi\|_{L^2_\alpha}^{\gamma}
\]
\[
= \left( \frac{\|C_\varphi\|_{L^2_\alpha}^{2\gamma}}{(1 - \lambda_j \lambda_i)^{\beta + 2}} - \frac{1}{(1 - |\varphi(\lambda_j)|^2)^{\beta + 2}} (1 - \lambda_j^\alpha |\varphi(\lambda_i)|^{\alpha + 2}) \right)^{\gamma - 1}
\]
\[
\times \sum_{k=0}^{\gamma - 1} \left( \|C_\varphi\|_{L^2_\alpha}^{2k} \right)^{\gamma - 1 - k}
\]
where \( c_k = (1 - \lambda_j \lambda_i)^{k\alpha + 2} \) and \( d_k = (1 - |\varphi(\lambda_j)|^2)^{\alpha + 2} \). Then
\[
\|C_\varphi\|_{L^2_\alpha}^{2\gamma} \leq \|C_\varphi\|_{L^2_\alpha} \leq \|C_\varphi\|_{L^2_\alpha}^{\gamma}
\]
\[
= \left( \frac{\|C_\varphi\|_{L^2_\alpha}^{2\gamma}}{(1 - \lambda_j^\alpha |\varphi(\lambda_i)|^{\alpha + 2})} - \frac{1}{(1 - |\varphi(\lambda_j)|^2)^{\beta + 2}} (1 - |\varphi(\lambda_j)|^2)^{\alpha + 2} \right)^{\gamma - 1}
\]
\[
\times \sum_{k=0}^{\gamma - 1} \left( \|C_\varphi\|_{L^2_\alpha}^{2k} \right)^{\gamma - 1 - k}
\]
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where \( a = (1 - |\lambda_2|^2)^k(\epsilon_\alpha + 2) \) and \( b = (1 - |\varphi(\lambda_2)|^2)^{(\epsilon_\alpha + 2)(\gamma - k - 1)} \).

Since the preceding equation holds for all \( i \) and \( j \), we obtain the following matrix equation:

\[
M \left( \| C_\varphi \|_{L^2_{\gamma_n}}, n, \beta \right) \\
= M \left( \| C_\varphi \|_{L^2_{\gamma_n}}, n, \alpha(\alpha + 2) \right) \\
\times \sum_{k=0}^{\gamma - 1} \left( \frac{\| C_\varphi \|_{L^2_{\gamma_n}}^{2k}}{e_k f_k} \right)^n_{i,j=1} (5)
\]

where \( e_k = (1 - \lambda_2 \lambda_1)^k(\epsilon_\alpha + 2) \) and \( f_k = (1 - \overline{\varphi(\lambda_2)} \varphi(\lambda_1))^{(\epsilon_\alpha + 2)(\gamma - k - 1)} \). This implies the matrix \( M \left( \| C_\varphi \|_{L^2_{\gamma_n}}, n, \alpha(\alpha + 2) \right) \) is positive semi-definite by Proposition 1.

Lemma 1, together with Proposition 2, dictates that every term in the matrix sum on the right-hand side of (5) is positive semi-definite, so the sum itself is positive semi-definite. Therefore Proposition 1 shows that \( M \left( \| C_\varphi \|_{L^2_{\gamma_n}}, n, \beta \right) \) must also be positive semi-definite.

Since this assertion holds for every natural number \( n \), we obtain by Proposition 1 that \( \| C_\varphi \|_{L^2_{\gamma_n}} \leq \| C_\varphi \|_{L^2_{\gamma_n}}^\beta \). Taking \( \alpha = 0, \alpha = -1 \) and \( \alpha = -2 \), we obtain the following corollaries.

**Corollary 1** Let \( \varphi \) be an analytic self-map of \( D \). Then

\[ \| C_\varphi \|_{L^2_{\gamma}} \leq \| C_\varphi \|_{H^2}^{\beta+2} , \]

whenever \( \beta \) is a non-negative integer. In particular, \( \| C_\varphi \|_{L^2} \leq \| C_\varphi \|_{H^2}^2 \).

**Corollary 2** Let \( \varphi \) be an analytic self-map of \( D \). Then

\[ \| C_\varphi \|_{L^2_{\gamma}} \leq \| C_\varphi \|_{L^2}^{(\beta+2)/2} , \]

whenever \( \beta \) is a positive even integer.

**Theorem 2** Take \( \beta \geq \epsilon_\alpha := (\alpha + 1)^2 - 1 \geq -1 \) and let \( \varphi \) be an analytic self-map of \( D \). Suppose that \( \gamma = (\beta + 2)/(\epsilon_\alpha + 2) \) is an integer. If \( C_\varphi \) is cosubnormal on \( L^2_{\epsilon_\alpha} \), then it is also cosubnormal on \( L^2_{\beta} \).

Cowen only stated this result for \( \alpha = -1 \), but an identical argument works for \( \alpha > -1 \). The proof makes use of Proposition 1 in a similar fashion to that of Theorem 1.

REFERENCES