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Shape preserving visualization of monotone data using a rational cubic ball function

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ABSTRACT: This paper deals with the problem of monotonicity-preserving curves of monotonic data. An alternative curve scheme using a piecewise rational cubic ball function is presented. The function involves three shape parameters in each subinterval. Data-dependent constraints are derived for a single shape parameter to preserve the shape of data while the other two are left free to modify the monotonic curve as desired. Several numerical examples are presented to show the effectiveness and capability of the proposed scheme. The scheme is C^2, flexible, simple, local, and economical.

KEYWORDS: interpolation, ball basis function, monotonicity

INTRODUCTION

Data visualization is the study of visual display representation of data and one of the factors that contributes to data visualization is shape preserving. Visualization of scientific data arises from scientific phenomena and complex functions to incorporate the inherited features of the data. The main goal of data visualization is to communicate information clearly and effectively through graphical means. The information needs to be presented in a way that is clear and easy to understand. These graphical representations of data plays a significant role not only in manufacturing of different product such as ship design, car modelling, aeroplane but also in the fields of engineering, military, education, art, medical research, image analysis, advertising and transport. There are three basic shape characteristics of data for curves and surfaces namely positivity, monotonicity and convexity. The aim of this paper is to preserve the meaning of underlying physical phenomena of monotonic data.

There are many physical situations of data that arise from different sciences and arts where they have a meaning when their values are monotone. Examples of monotonic data are approximations of couple and quasi couples in statistics, approximations of potential functions in physical and chemical systems and dose response curves in biochemistry and pharmacology, level of uric acid in gout patients, data generated from stress and strain of materials, graphical display of Newton's law of cooling and medical diagnosis and economic forecasting.

Since ordinary spline interpolating curve scheme does not preserve the shape feature of monotonicity of monotone data, some alteration are needed to preserve the shape of such curves. Normally, ordinary curve scheme used Bézier-Bernstein basis functions for C^1 and C^2 continuity of smoothness of interpolation. For this reason, we propose a new scheme using ball basis functions which involves three shape parameters and the smoothness of the interpolation is C^2 continuity. The scheme only deals with monotonicity preservation of curves through monotone data

In recent years, problem of monotonicity on shape preservation has been dealt by many authors. Abbas et al[1] developed monotonicity-preserving interpolation with three shape parameters to maintain the shape of monotonic data. The authors also derived data dependent conditions. Abbas[2] developed a C^1 and C^2 piecewise cubic rational monotonicity-preserving interpolation and introduced a rational cubic function with three shape parameters that provide freedom to designers to modify the curves for interactive design. Abbas et al[3] considered a C^1 piecewise rational cubic function with three shape parameters in each interval to preserve the monotonicity of data. Butt[4] inserted additional knots rather than using certain choices of slope in order to produce stiffness of cubic Hermite interpolation. Duan et al[5] used function values to develop rational interpolation. The authors also discussed how to constraint the interpolating curves in order to lie above, below or in between two lines. Fiorot and Tabka[6] introduced C^2 cubic polynomial spline for preserving monotonicity or convexity. The authors also introduced a new method to get the values of derivative parameters d_i' s through system of linear equations. Lamberti and Manni[7] focused on parametric cubic to approximate and explore the order of a global C^2 shape preserving interpolating function. They used tension parameters to control the shape of curve. The authors also constructed the necessary and sufficient conditions for convexity. Wang and Tan[8] developed a C^2 piecewise rational quartic spline function with two shape parameters. Piah and Unsworth[9] developed improved sufficient conditions of a Bernstein-Bézier quartic rational with quartic numerator and linear denominator and considered only single shape parameter to preserve the shape of monotone data. The authors also improved the region of monotonicity inspired by Wang and Tan[8] without considering error estimation for interpolating. Sarfraz et al[10] developed rational cubic spline to provide smoothness of positivity, monotonicity and convexity curves. The authors also introduced two families of parameters in order to control the shape of curve.

In this work, an alternative curve scheme using piecewise rational cubic ball function is developed to ensure C^2 continuity. The function involves three shape parameters u_i, v_i and w_i to preserve the shape of the data. There are two kind of shape parameters u_i, v_i are free parameters which can be used to adjust that shape of the curve and w_i plays the role of shape parameter automatically to ensure monotone data are preserved. This work is an involvement of reviews by many authors. The new scheme has the following useful and benefits features.

(i) It provides a C^2 degree of smoothness, while[1,4] the degree of smoothness is C^1.

(ii) No extra knots are needed as in where[4] the scheme is done by inserting extra knots between any two knots in the interval.

(iii) Based on experimental results the present scheme is flexible, simple, local and economical. It is found that the generated curve is visually pleasing as compared to the existing schemes in Refs. 1, 8–10.

(iv) The present scheme is appropriate for uniform and non-uniform spaced data while the scheme in Ref. 4 only works for uniform spaced data.

(v) In this paper, a new method to compute derivative parameters using tridiagonal system of linear equations is much more efficient than, for instance, solving the three systems of linear equations in Ref. 5.

(vi) In the proposed scheme, users are allowed to refine the curve by introducing free parameters which can be used freely to generate visually better pleasant curve but in Ref. 9 the authors proposed scheme does not allow the user to refine the shape of curves.

(vii) This paper deals with the problem related to rational cubic ball function (cubic/cubic) to generate monotonic curve through given monotonic data while in Wang and Tan[8] and Piah and Unsworth[9] they used quartic over linear Bernstein-Bézier function to ensure C^2 continuity.

The remainder of the paper is organized as follows. The second section introduces a rational cubic ball interpolant and determination of derivatives is discussed in the third section. The fourth section discusses rational cubic ball function in terms of preserving monotonic data and how to generate C^2 piecewise interpolants. The outputs obtained from three test cases are presented in the fifth section.

RATIONAL CUBIC BALL FUNCTION

Let $\{(t_i, y_i), i = 0, 1, 2, \ldots, n\}$ be a given set of data points. It is defined over the interval $[a, b]$ such that $a = t_0 < t_1 < t_2 < \ldots < t_n = b$. A piecewise rational cubic ball function is defined in each subinterval $I_i = [t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ as

$$B(t) \equiv B_i(\varphi) = \frac{s_i(\varphi)}{r_i(\varphi)} \tag{1}$$

where

$$\begin{aligned} s_i(\varphi) &= \xi_0(1-\varphi)^2 + \xi_1\varphi(1-\varphi)^2 \\ &\quad + \xi_2\varphi^2(1-\varphi) + \xi_3\varphi^2, \\ r_i(\varphi) &= u_i(1-\varphi)^2 + \alpha_i\varphi(1-\varphi)^2 \\ &\quad + \beta_i\varphi^2(1-\varphi) + v_i\varphi^2, \end{aligned} \tag{2}$$

and

$$\begin{aligned} h_i &= t_{i+1} - t_i, \\ \Delta_i &= \frac{(y_{i+1} - y_i)}{h_i}, \\ \varphi &= \frac{t - t_i}{h_i}, \qquad \varphi \in [0, 1]. \end{aligned} \tag{3}$$

Here, both numerator and denominator are the usual cubic ball polynomial instead of the usual cubic Bernstein-Bézier polynomials and u_i, α_i, β_i and v_i are non-zero shape parameters. According to (1), the denominator is non-zero. By collecting the middle values of weights and defined as $\alpha_i = \beta_i = u_i + v_i + w_i \geqslant 0$, $i = 0, 1, 2, \ldots, n-1$, equation (1) can be written in terms of the three shape parameters in which two are free shape parameters and w_i is a constrained shape parameter. A rational cubic ball function with three shape parameter (1) can be rewritten as

$$B(t) \equiv B_i(\varphi) = \frac{p_i(\varphi)}{q_i(\varphi)} \tag{4}$$

where

$$\begin{aligned} p_i(\varphi) &= \xi_0(1-\varphi)^2 + \xi_1\varphi(1-\varphi)^2 \\ &\quad + \xi_2\varphi^2(1-\varphi) + \xi_3\varphi^2, \\ q_i(\varphi) &= u_i(1-\varphi)^2 \\ &\quad + (u_i + w_i + v_i)(1-\varphi) + v_i\varphi^2. \end{aligned} \tag{5}$$

A rational cubic ball function (4) satisfies the following properties to ensure C^2 continuity:

$$\begin{aligned} &B(t_i) = y_i, \quad B(t_{i+1}) = y_{i+1}, \\ &B'(t_i) = d_i, \quad B'(t_{i+1}) = d_{i+1}, \\ &B''(t_i^+) = B''(t_i^-), \quad i = 1,\ 2, \ldots, n-1, \end{aligned} \tag{6}$$

where $B'(t_i)$ and $B''(t_i)$ denote the first and second derivatives with respect to t, respectively, and d_i denotes the values of the derivative (tangents) at the knots t_i. From (6), it is easy to get the following system of linear equations and values of unknowns ξ_i, $i = 0, 1, 2, 3$

$$\eta_i d_{i-1} + \kappa_i d_i + \mu_i d_{i+1} = \zeta_i \tag{7}$$

with

$$\begin{aligned} \eta_i &= u_{i-1}u_ih_i, \\ \kappa_i &= (u_{i-1} + v_{i-1} + w_{i-1})u_ih_i \\ &\quad + (u_i + v_i + w_i)v_{i-1}h_{i-1}, \\ \mu_i &= v_{i-1}v_ih_{i-1}, \\ \zeta_i &= (2u_{i-1} + v_{i-1} + w_{i-1})\Delta_{i-1}u_ih_i \\ &\quad + (u_i + 2v_i + w_i)\Delta_i v_{i-1}h_{i-1}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \xi_0 &= u_iy_i, \\ \xi_1 &= (u_i + v_i + w_i)y_i + u_ih_id_i, \\ \xi_2 &= (u_i + v_i + w_i)y_{i+1} - v_ih_id_{i+1}, \\ \xi_3 &= v_iy_{i+1}. \end{aligned} \tag{9}$$

When the values of shape parameters are $u_i = 1$, $v_i = 1$ and $w_i = 0$ in each subinterval $I_i = [t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$, the rational cubic ball function reduces to a non-rational cubic ball function like cubic Hermite spline. Variation of values in u_i 's and v_i 's are used to control the curve and as a result the curves becomes tight or loose in every segment.

DETERMINATION OF DERIVATIVES

In most applications, the derivative values or tangents d_i are calculated from the given data points because d_i are not given directly. The purpose of using derivatives values d_i is for the smoothness of curve. In equation (7), the system of linear equations is a dominant tridiagonal system and has a unique solution for all positive shape parameters. The system produces $(n-2)$ linear equations and provides the values of unknown derivative parameter d_i, $i = 1, 2, \ldots, n-1$. The system can be solved for the values of derivative parameters d_i's using the LU decomposition method. If the end points derivatives d_1 and d_2 are not given, then it can be derived using the following approximations from Abbas[2]:

$$B'(t_0) = d_0, \quad B'(t_n) = d_n.$$

MONOTONICITY-PRESERVING C^2 RATIONAL CUBIC BALL INTERPOLATION

Let us assume (t_i, y_i), $i = 0, 1, 2, \ldots, n$ as a given increasing set of monotonic data such that $t_0 < t_1 < t_2 < \ldots < t_n$, i.e.,

$$y_i \leqslant y_{i+1}, \qquad i = 0, 1, 2, \ldots, n-1.$$

Similarly for monotonically decreasing data, or equivalently

$$\Delta_i \geqslant 0, \qquad i = 0, 1, 2, \ldots, n-1.$$

The derivative parameters, $d_i \geqslant 0$ (for monotonically increasing data) and $d_i \leqslant 0$ (for monotone decreasing data). There are two cases of monotonicity for increasing data

Case 1: If $\Delta_i = 0$, then the values of derivatives are $d_i = d_{i+1} = 0$ and $B(t)$ reduces to

$$B_i(t) = y_i, \quad \forall t \in [t_i, t_{i+1}], \quad i = 0, 1, 2, \ldots, n-1,$$

i.e., the interpolant is automatically monotonic.

Case 2: If $\Delta_i \geqslant 0$, the interpolant $B(t)$ preserves monotonicity when $B_i'(t) > 0$ for all $t \in [t_i, t_{i+1}]$. The simpler form of $B_i'(t)$ can be shown as

$$B_i'(t) = \frac{k_i(\varphi)}{(r_i(\varphi))^2}$$

where

$$\begin{aligned} k_i(\varphi) &= A_{1,i}(1-\varphi)^3 + A_{2,i}\varphi(1-\varphi)^3 \\ &\quad + A_{3,i}\varphi^2(1-\varphi)^2 + A_{4,i}\varphi^3(1-\varphi) \\ &\quad + A_{5,i}\varphi^3 \end{aligned} \tag{10}$$

with

$$A_{1,i} = u_i^2 d_i,$$
$$A_{2,i} = 2u_i(u_i + 2v_i + w_i)\Delta_i - (u_i^2 d_i + 2d_i u_i v_i),$$
$$A_{3,i} = 2[(u_i^2 + 4u_i v_i + v_i) + 3(u_i + v_i)w_i + w_i^2]\Delta_i - \left[v_i d_{i+1}(2u_i + v_i + w_i) + u_i d_i(u_i + 2v_i + w_i)\right],$$
$$A_{4,i} = 2v_i(2u_i + v_i + w_i)\Delta_i - (v_i^2 d_{i+1} + 2d_i u_i v_i),$$
$$A_{5,i} = v_i^2 d_{i+1}.$$

The necessary conditions for monotonically preserving curve are

$$d_i \geqslant 0, \quad u_i > 0, \quad v_i > 0, \quad w_i \geqslant 0. \tag{11}$$

From the conditions in (11), it is clear that $A_{1,i}$ and $A_{5,i}$ are positive. $A_{2,i} > 0$ if

$$w_i > \frac{u_i d_i + 2d_{i+1} v_i}{2\Delta_i}, \tag{12}$$

$A_{4,i} > 0$ if

$$w_i > \frac{2u_i d_i + d_{i+1} v_i}{2\Delta_i}, \tag{13}$$

and $A_{3,i} > 0$ if

$$w_i > \frac{u_i d_i + d_{i+1} v_i}{\Delta_i}. \tag{14}$$

The constraint on w_i in (14) is only competent and rationally to choose for monotonicity because $A_{2,i}$, $A_{4,i}$ are also positive. The above conditions can be summarized as

$$\begin{gathered} u_i > 0, v_i > 0 \\ w_i > \max\left\{0, \frac{u_i d_i + d_{i+1} v_i}{\Delta_i}\right\}. \end{gathered} \tag{15}$$

The above result can be re-summarized as

$$\begin{gathered} u_i > 0, v_i > 0 \\ w_i = m_i + \max\left\{0, \frac{u_i d_i + d_{i+1} v_i}{\Delta_i}\right\}. \end{gathered} \tag{16}$$

for some $m_i > 0$.

Theorem 1 *A rational cubic ball function* (4) *preserves the C^2 monotonic curve of monotonic data over the interval $[t_i, t_{i+1}]$ if and only if shape parameters u_i, v_i and w_i satisfy* (16).

Example 1 The data were collected from a cricket match where the total score of the team at different number of balls was recorded in Table 1. The t-values are number of balls and y-values are numbers of scores. The curve in Fig. 1 is drawn by the non-rational cubic ball function (4) with shape parameters $u_i = 1$, $v_i = 1$, and $w_i = 0$. Generally, the curve is smooth but the monotonicity is lost which does not make any sense physically. On the other hand, Figs. 2 and 3 are generated by monotonicity-preserving C^2 rational cubic ball interpolant with different values of shape parameters. Fig. 4 is produced by a built in MATLAB program PCHIP (Piecewise cubic Hermite

Table 1 Monotone data set obtained from cricket match.

i	1	2	3	4	5	6	7	8	9	10	11
t_i	1	2	6	8	15	25	40	50	62	65	66
y_i	1	1	2	4	6	6	10	10	13	18	20

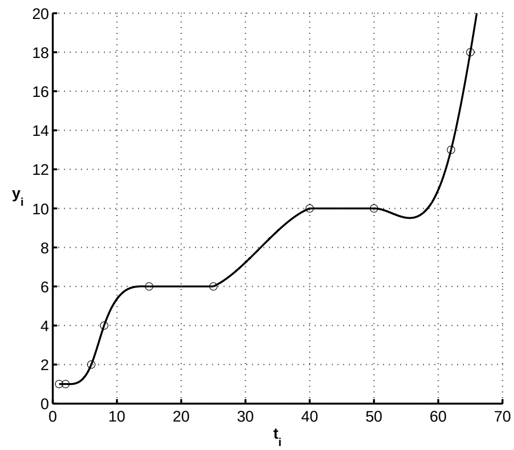

Fig. 1 Non-rational cubic ball curve does not preserve monotonicity using $u_i = v_i = 1$ and $w_i = 0$.

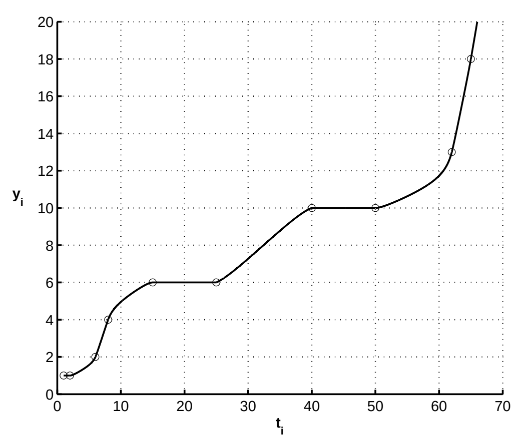

Fig. 2 Monotonicity-preserving C^2 rational cubic ball curve with $u_i = v_i = 0.5$.

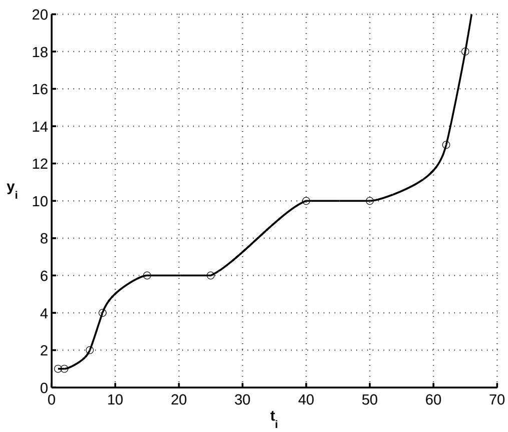

Fig. 3 Monotonicity-preserving C^2 rational cubic ball curve with $u_i = v_i = 2.5$.

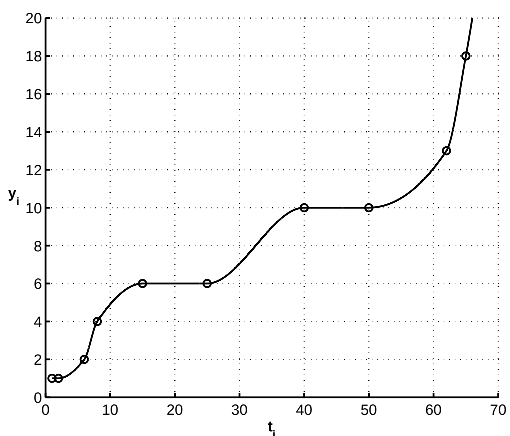

Fig. 4 PCHIP curve.

Interpolating Polynomial). It is easy to see that Figs. 2 and 3 are visually pleasing and smooth as compared to Fig. 4.

NUMERICAL EXAMPLES

Example 2 A monotonic data[3] set in Table 2 is the experimental results of the Great Northern beans. Chemical solutions are made by mixing chemical flake (KOH) and distilled water with a pH of 8.5 to water the beans. After 40 days, the effects of the solution on bean plants can be seen by removing and weighing bean plants from the vat. The t-values represent the days and y-values indicate the height of the beans. One can observed that the examined data is monotone. Fig. 5 is produced by the non-rational cubic ball function (4) that does not preserve monotonicity of the data. Figs. 6 and 7 are produced from the same monotonic data using rational cubic ball interpolant. Fig. 7 improves the smoothness of the curves in Fig. 6 and PCHIP curve in Fig. 8.

Example 3 An Akima monotone data set taken in Table 3 is generated by a Piah and Unsworth[9]. Fig. 9

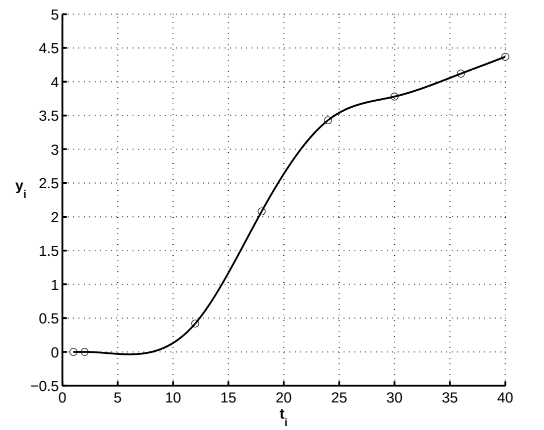

Fig. 5 Non-rational cubic ball curve using $u_i = v_i = 1$ and $w_i = 0$.

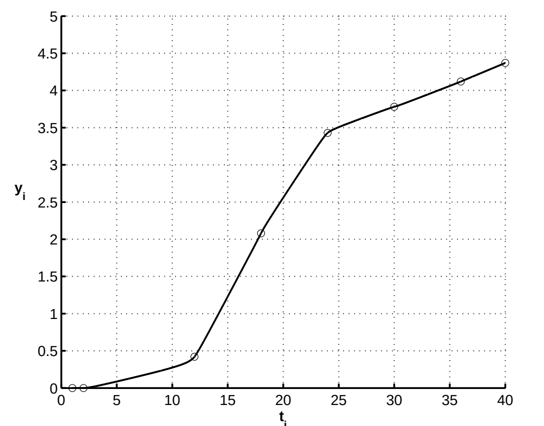

Fig. 6 Monotonic C^2 rational cubic ball curve with $u_i = v_i = 0.1$.

Table 2 2D monotone data set.

i	1	2	3	4	5	6	7	8
t_i	1	2	12	18	24	30	36	40
y_i	0	0	0.42	2.08	3.43	3.78	4.12	4.37

Table 3 Akima's data set.

i	1	2	3	4	5	6	7	8	9	10	11
t_i	0	2	3	5	6	8	9	11	12	14	15
y_i	10	10	10	10	10	10	10.5	15	50	60	85

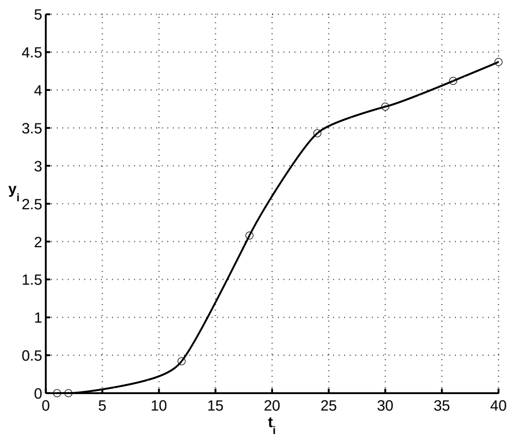

Fig. 7 Monotonic C^2 rational cubic ball curve with $u_i = v_i = 2.5$.

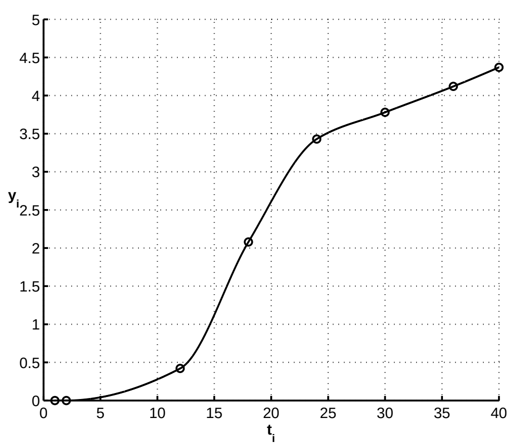

Fig. 8 PCHIP curve.

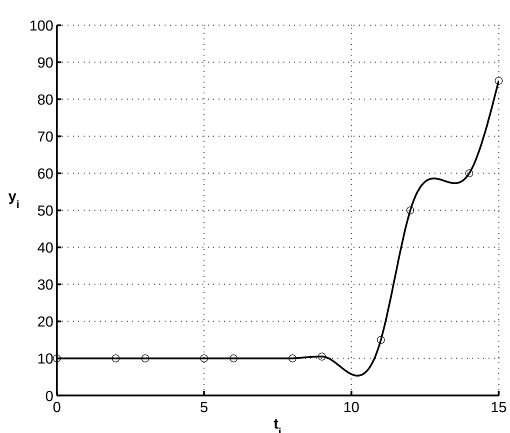

Fig. 9 Non-rational cubic ball curve using $u_i = v_i = 1$ and $w_i = 0$.

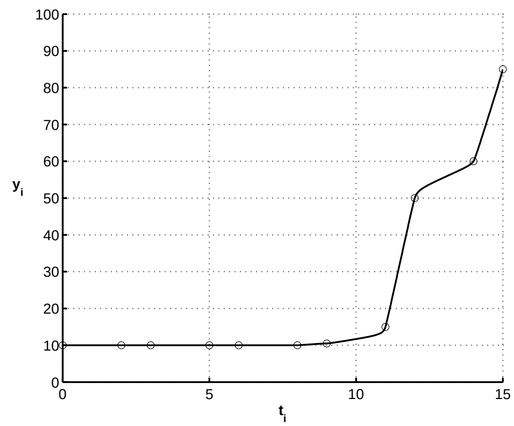

Fig. 10 Monotonicity-preserving C^2 rational cubic ball curve with $u_i = v_i = 0.1$.

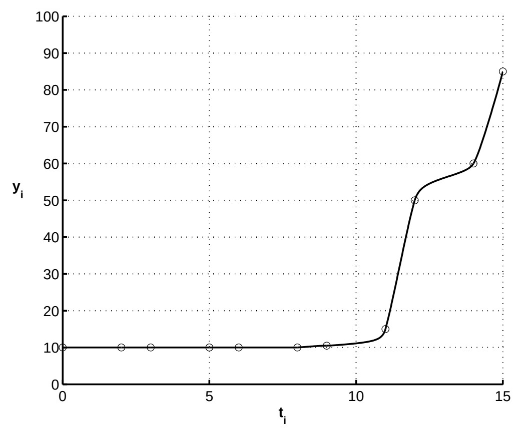

Fig. 11 Monotonicity-preserving C^2 rational cubic ball interpolant with $u_i = v_i = 1.5$.

is generated by a non-rational cubic ball function. On the other hand, Figs. 10 and 11 are generated by proposed monotone rational cubic ball interpolant using different values of shape parameters to preserve the shape of monotonic data. The curve in Fig. 11 is smoother than PCHIP curve in Fig. 12.

CONCLUDING REMARKS AND SUGGESTIONS

In this paper, we propose and analyse the problem of shape preservation of monotonic data. We have developed a rational cubic ball function with three shape parameters. The proposed scheme is suitable for monotonicity-preserving problems in which only data points are given. In this scheme, there is no necessity to insert of extra points. Moreover, the scheme calculates derivative parameters by solving a single system of linear equations which is flexible,

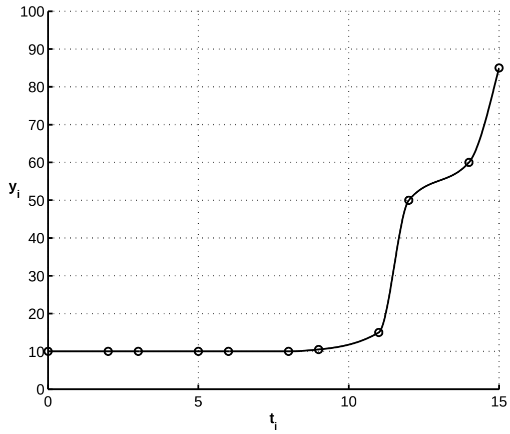

Fig. 12 PCHIP curve.

simple and economical, as compared to Fiorot and Tabka[5] who calculated the derivative parameters by solving three tridiagonal systems of linear equations.

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