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Blow-up in non-autonomous semilinear pseudoparabolic equations

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ABSTRACT: We study the blow-up property for weak solutions to the Cauchy problem of non-autonomous semilinear pseudoparabolic equations. Given the growth bound of the non-autonomous coefficient, the Fujita-type critical exponent is obtained.

KEYWORDS: critical exponent, test function method, global solutions

INTRODUCTION

We study non-negative weak solutions $u = u(x,t)$ of the Cauchy problem

$$\begin{cases} \partial_t u - \Delta\partial_t u = \Delta u + V(x)u^p, & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n \end{cases} \tag{1}$$

where $p > 1$ is a constant and V, u_0 are given non-negative functions. This partial differential equation (PDE) is called a *pseudoparabolic equation*[1–4]. It is semilinear and non-autonomous owing to the coefficient $V(x)$ on the right-hand side. Nonlinear pseudoparabolic equations have been proposed to model many physical systems; for instance, the non-steady flow of second-order fluids in one space dimension[5], seepage of homogeneous fluids through fissured rock[6], heat conduction involving two temperatures[7].

The equation (1) is also closely related with the following non-autonomous semilinear heat equation

$$\partial_t u = \Delta u + V(x)u^p$$

and the latter has been widely investigated by many authors[8]. In the case $V(x) \equiv 1$, the problem (1) becomes autonomous and was investigated by Cao et al[9]. In their paper, the existence of mild solutions, which are also weak solutions, was established. Using the energy method, the authors obtained the critical exponent of the problem, denoted by p_c, for the class of classical solutions:

$$p_c = 1 + \frac{2}{n}. \tag{2}$$

This means that if $1 < p \leqslant p_c$ then every nontrivial non-negative solution to the problem blows up in some finite time T_0, i.e., $\lim_{t\to T_0^-} \|u(\cdot,t)\|_{L^\infty} = \infty$. On the other hand, if $p > p_c$ there are both blowing-up solutions (for sufficiently large u_0) and global-in-time solutions (for sufficiently small u_0). Even though the blow-up phenomenon has played an important role in PDE theory[10,11], the blowing-up problem of (1) for non-constant V, however, remains open.

In this study, the Cauchy problem (1) with a broader class of functions V is considered and the critical exponent analogous to (2) is obtained. The energy method does not seem to work for the weak solution in the case where V is non-constant and therefore a new approach is needed. The technique employed here is the test function (or nonlinear capacity) method[12]. Other important related questions (e.g., existence, uniqueness, regularity, and large-time asymptotic) will be addressed in our forthcoming papers.

PRELIMINARIES

Let $Q_T = \mathbb{R}^n \times [0,T)$ and $Q_\infty = \mathbb{R}^n \times [0,\infty)$.

Lemma 1 (Folland[13]) *Let (X,μ) be a measure space and $1 \leqslant p,q \leqslant \infty$.*
(i) *For all $a,b \geqslant 0$ and $\lambda \in [0,1]$, we have*

$$a^\lambda b^{1-\lambda} \leqslant \lambda a + (1-\lambda)b.$$

(ii) *If $f \in L^p(X)$, $g \in L^q(X)$ where $1/p + 1/q = 1$, then $h = fg \in L^1(X)$ and*

$$\|h\|_{L^1(X)} \leqslant \|f\|_{L^p(X)} \|g\|_{L^q(X)}.$$

The solutions of (1) considered in this paper are weak solutions which are defined as follows.

Definition 1 A function u is called a *weak* (or *distributional*) *solution* to the problem (1) on $I = [0,T)$ provided
(i) $u \in C(I; L^1_{\text{loc}}(\mathbb{R}^n))$, $Vu^p \in L^1_{\text{loc}}(I; L^1_{\text{loc}}(\mathbb{R}^n))$,
(ii) for all $\varphi \in C^3_c(Q_T)$, the following identity holds:

$$\iint_{Q_T} u(\partial_t\varphi - \Delta\partial_t\varphi + \Delta\varphi) + \iint_{Q_T} Vu^p\varphi = \int_{\mathbb{R}^n} u_0(\Delta\varphi - \varphi)|_{t=0} \quad (3)$$

If (i) and (ii) are true with $T = \infty$, then u is called a *global weak solution*.

The following lemma[12] will be used to construct the test functions needed below. For the completeness of the paper, the proof is provided.

Lemma 2 *For any $q \in (1,\infty)$, there is a C^3 function $\phi : \mathbb{R} \to [0,1]$ with $\phi(s) = 1$ if $s \leqslant 1$, $0 \leqslant \phi(s) \leqslant 1$ if $1 \leqslant s \leqslant 2$, $\phi(s) = 0$ if $s \geqslant 2$, and*

$$|\phi'(s)|^q + |\phi''(s)|^q + |\phi'''(s)|^q \leqslant C_\phi \phi(s)^{q-1}$$

for all $s \in \mathbb{R}$, for some constant $C_\phi > 0$.

Proof: Choose a function $\zeta \in C^3(\mathbb{R}, [0,1])$ with

$$\zeta(s) \begin{cases} = 1, & s \leqslant 1, \\ \in (0,1), & 1 < s < 2, \\ = 0, & s \geqslant 2 \end{cases}$$

and let $\phi(s) = \zeta(s)^{3q}$. Then

$$\begin{aligned} |\phi'|^q &= |3q\zeta'|^q \zeta^{(3q-1)q} \leqslant C\phi^{q-1} \\ |\phi''|^q &= \left|3q(3q-1)(\zeta')^2 + 3q\zeta\zeta''\right|^q \zeta^{(3q-2)q} \\ &\leqslant C\phi^{q-1} \\ |\phi'''|^q &= |3q(3q-1)(3q-2)(\zeta')^3 \\ &\quad + 3(3q)(3q-1)\zeta\zeta'\zeta'' + 3q\zeta^2\zeta'''|^q \zeta^{(3q-3)q} \\ &\leqslant C\phi^{q-1} \end{aligned}$$

because $(3q-i)q \geqslant 3q(q-1)$ for $i = 1,2,3$ and $0 \leqslant \zeta \leqslant 1$ where $C > 0$ is a constant depending only on q, $\|\zeta'\|_{L^\infty}$, $\|\zeta''\|_{L^\infty}$, and $\|\zeta'''\|_{L^\infty}$. □

Remark 1 Generally, for any $q \in (1,\infty)$ and $k \in \mathbb{N}$, there is $\phi \in C^k([0,\infty), [0,1])$ satisfying

$$\sum_{i=1}^{k} \left|\phi^{(i)}(s)\right|^q \leqslant C\phi(s)^{q-1} \quad \forall\, s \geqslant 0$$

for some constant C.

MAIN RESULTS

We will consider the class of functions V in (1) that satisfies the following assumption.

Assumption. The function $V(x)$ has an order of growth of at least $\sigma > -2$, in the sense that there exists $x_0 \in \mathbb{R}^n$ and a constant $c_0 > 0$ such that

$$V(x) \geqslant c_0 |x - x_0|^\sigma \quad (4)$$

for almost every $x \in \mathbb{R}^n$.

The proof below is valid for arbitrary x_0. However, for simplicity of presentation and without loss of generality, we let $x_0 = 0$.

Theorem 1 *Assume (4) on V and let $1 + (\sigma^+)/n < p \leqslant 1 + (\sigma+2)/n$ where $\sigma^+ = \max\{0,\sigma\}$. If $0 \leqslant u_0 \in L^1(\mathbb{R}^n)$ with $\|u_0\|_{L^1(\mathbb{R}^n)} > 0$, then there is no nontrivial, non-negative global weak solution u to the problem (1).*

Proof: The theorem will be proved by contradiction. We therefore assume the contrary that the problem (1) admits a non-trivial, global weak solution u. We divide the proof into 5 steps.

Step 1. Define the operator

$$\mathcal{A}\varphi := \partial_t\varphi - \Delta\partial_t\varphi + \Delta\varphi,$$

for all test functions $\varphi \in C^3_c(Q_\infty)$. Then u satisfies for all φ the identity

$$\iint_{\operatorname{supp}\mathcal{A}\varphi} u\mathcal{A}\varphi + \iint_{\operatorname{supp}\varphi} Vu^p\varphi = \int_{\mathbb{R}^n} u_0(\Delta\varphi - \varphi)|_{t=0}. \quad (5)$$

Choose φ to satisfy $0 \leqslant \varphi \leqslant 1$ and $\varphi|_{\operatorname{supp}\mathcal{A}\varphi} > 0$ a.e. so that $\operatorname{supp}\mathcal{A}\varphi \subset \operatorname{supp}\varphi$.

Let $K = \operatorname{supp}\varphi$ and $K' = \operatorname{supp}\mathcal{A}\varphi$. By the Hölder and Young inequalities, we have

$$\begin{aligned} &\iint_{K'} |u\mathcal{A}\varphi| \\ &\leqslant \left(\iint_{K'} Vu^p\varphi\right)^{1/p} \left(\iint_{K'} \frac{|\mathcal{A}\varphi|^q}{(V\varphi)^{q-1}}\right)^{1/q} \\ &\leqslant \frac{1}{p}\iint_K Vu^p\varphi + \frac{1}{q}\iint_{K'} \frac{|\mathcal{A}\varphi|^q}{(V\varphi)^{q-1}}, \end{aligned} \quad (6)$$

where $q = (p/(p-1)) \in (1,\infty)$. Combining (5), (6) with the assumption $V(x) \geqslant c_0|x|^\sigma$ a.e. yields the estimate

$$\iint_K Vu^p\varphi \leqslant \frac{1}{c_0^{q-1}} \iint_{K'} \frac{|\mathcal{A}\varphi|^q}{(|x|^\sigma\varphi)^{q-1}} + q\int_{\mathbb{R}^n} u_0(\Delta\varphi - \varphi)|_{t=0}. \quad (7)$$

Step 2. Let us further specify φ. Fix a function ϕ satisfying Lemma 2. For $R \gg 1$, define φ by

$$\varphi(x,t) = \phi\left(\frac{t+|x|^2}{R^2}\right).$$

Below, the following rescaling variables will be used: $\tau = t/R^2$, $\xi = |x|/R$, and $s = \tau + \xi^2$. As subsets in the ξt-plane, $\{(\xi,\tau) : 1 \leqslant \tau+\xi^2 \leqslant 2\} = K' \subset K$ and $K \subset [0,\sqrt{2}] \times [0,2]$. Direct computation shows that

$$\begin{aligned}
\partial_t \varphi &= \frac{1}{R^2}\phi', \\
\Delta\varphi &= \frac{1}{R^2}\Big(4\xi^2\phi'' + 2n\phi'\Big), \\
\partial_t \Delta\varphi &= \frac{1}{R^4}(4\xi^2\phi''' + 2n\phi''), \quad \text{and} \\
\mathcal{A}\varphi &= \frac{2n+1}{R^2}\phi' + \frac{4\xi^2R^2 - 2n}{R^4}\phi'' - \frac{4\xi^2}{R^4}\phi'''.
\end{aligned}$$

In particular, we have

$$|\Delta\varphi|_{t=0} \leqslant \frac{C_0}{R^2}, \tag{8}$$

where $C_0 = 8\|\phi''\|_{L^\infty} + 2n\|\phi'\|_{L^\infty}$. Using the fact that φ satisfies Lemma 2, we obtain, for all $R \gg 1$, that

$$\begin{aligned}
|\mathcal{A}\varphi|^q &= \left|\frac{2n+1}{R^2}\phi' + \frac{4\xi^2R^2 - 2n}{R^4}\phi'' - \frac{4\xi^2}{R^4}\phi'''\right|^q, \\
&\leqslant \frac{C_{n,q}}{R^{2q}}(|\phi'|^q + |\phi''|^q + |\phi'''|^q), \\
&\leqslant \frac{C_1}{R^{2q}}\varphi^{q-1},
\end{aligned}$$

where $C_1 = C_{n,q}C_\phi$.

Step 3. We perform the polar integration $dxdt = R^{n+2}\xi^{n-1}\mathrm{d}\xi d\omega \mathrm{d}\tau$ to get that

$$\begin{aligned}
&\iint_{K'} \frac{|\mathcal{A}\varphi|^q}{(|x|^\sigma\varphi)^{q-1}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \iiint_{K'} \frac{C_1}{(R\xi)^{\sigma(q-1)}R^{2q}} R^{n+2}\xi^{n-1}\,\mathrm{d}\xi\,\mathrm{d}\omega\,\mathrm{d}\tau \\
&\leqslant \frac{C_1\omega_n}{R^e}\iint_{1\leqslant\tau+\xi^2\leqslant 2} \xi^{\alpha-1}\,\mathrm{d}\xi\,\mathrm{d}\tau, \quad (\omega_n = |S^{n-1}|) \\
&\leqslant \frac{2C_1\omega_n}{R^e}\int_0^{\sqrt{2}} \xi^{\alpha-1}\,\mathrm{d}\xi,
\end{aligned}$$

where $\alpha = (n/(p-1))(p-1-\sigma/n)$ and $e = (n/(p-1))(1+(\sigma+2)/n - p)$. Since $p > 1 + ((\sigma^+)/n) \geqslant 1 + \sigma/n$, we have $\alpha > 0$. Hence

$$\iint_{K'} \frac{|\mathcal{A}\varphi|^q}{(|x|^\sigma\varphi)^{q-1}} \leqslant \frac{M}{R^e}, \tag{9}$$

where $M = 2^{1+\alpha/2}C_1\omega_n$. Plugging (8) and (9) in (7) yields

$$\begin{aligned}
&\iint_K Vu^p\varphi \\
&\qquad \leqslant \frac{Mc_0^{1-q}}{R^e} + q\int_{\mathbb{R}^n} u_0(\Delta\varphi - \varphi)|_{t=0} \qquad (10) \\
&\qquad \leqslant \frac{Mc_0^{1-q}}{R^e} + q\int_{\mathbb{R}^n} u_0|\Delta\varphi|_{t=0} \\
&\qquad \leqslant \frac{Mc_0^{1-q}}{R^e} + \frac{qC_0}{R^2}\|u_0\|_{L^1}. \qquad (11)
\end{aligned}$$

Step 4. Now consider the case $p < 1 + (\sigma+2)/n$. It is obvious that $e > 0$. Since $\varphi \equiv 1$ on $\{(x,t) : 0 \leqslant t + |x|^2 \leqslant R^2\} \subset K$, it follows that

$$\iint_{0\leqslant t+|x|^2\leqslant R^2} Vu^p \leqslant \frac{Mc_0^{1-q}}{R^e} + \frac{qC_0}{R^2}\|u_0\|_{L^1},$$

for all $R \gg 1$. As $e > 0$, the right-hand side converges to 0 as $R \to \infty$. Hence $\iint_{Q_\infty} Vu^p = 0$ which implies $u \equiv 0$ contradicting the non-triviality of u.

Step 5. For the case $p = 1 + (\sigma+2)/n$, the right-hand side of (11) is bounded as $R \to \infty$. Hence

$$\iint_{Q_\infty} Vu^p\,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Therefore Vu^p is integrable. By (5), (8), (9), and Hölder's inequality, we have

$$\begin{aligned}
&\iint_K Vu^p\varphi \\
&\quad \leqslant \iint_{K'} u|\mathcal{A}\varphi| + \frac{qC_0}{R^2}\|u_0\|_{L^1} \\
&\quad \leqslant \left(\iint_{K'} Vu^p\varphi\right)^{1/p}\left(\iint_{K'}\frac{|\mathcal{A}\varphi|^q}{(V\varphi)^{q-1}}\right)^{1/q} \\
&\qquad\quad + \frac{qC_0}{R^2}\|u_0\|_{L^1} \\
&\quad \leqslant \frac{M^{1/q}}{c_0^{q-1}}\left(\iint_{K'} Vu^p\varphi\right)^{1/p} + \frac{qC_0}{R^2}\|u_0\|_{L^1} \\
&\quad \leqslant \frac{M^{1/q}}{c_0^{q-1}}\left(\iint_{K'} Vu^p\right)^{1/p} + \frac{qC_0}{R^2}\|u_0\|_{L^1}. \qquad (12)
\end{aligned}$$

Since $K' \subset \{(x,t) : R^2 \leqslant t^2 + |x|^2 \leqslant 2R^2\}$, the integrability of Vu^p implies $\iint_{K'} Vu^p \to 0$ as $R \to \infty$. Therefore, by letting $R \to \infty$, we obtain from (12) that

$$\iint_{Q_\infty} Vu^p = 0,$$

which implies $u \equiv 0$, and again a contradiction. □

For the next result, we will show that when $p > 1+(\sigma+2)/n$, weak solutions to the Cauchy problem (1) blow up in a finite time if u_0 is large enough.

Theorem 2 *Let $p > 1+(\sigma+2)/n$. If $u_0 \in L^1(\mathbb{R}^n)$ is sufficiently large in the sense that there exists $R_0 \geqslant C_0^{1/2}$, where C_0 is given in (8) depending on ϕ from Lemma 2, such that*

$$\int_{B_{R_0}(0)} u_0(x) \geqslant \max\left\{\tfrac{3}{4}\|u_0\|_{L^1}, \frac{4Mc_0^{1-q}}{qR_0^e}\right\},$$

then every weak solution u to the Cauchy problem (1) blows up in a finite time.

Proof: Again, we will prove by contradiction and therefore assume that the global weak solution u exists. Set $R = R_0 \gg 1$ in the proof of the preceding theorem. Since $R_0 \geqslant C_0^{1/2}$, (8) can be reduced to

$$|\Delta\varphi(x,0)| \leqslant 1.$$

In (10), which is true for all cases of $p > 1+(\sigma^+)/n$, the second term on the right-hand side can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^n} & u_0(x)[\Delta\varphi(x,0)-\varphi(x,0)]\,\mathrm{d}x \\ &\leqslant \int_{|x|\geqslant R_0} u_0(x) - \int_{|x|\leqslant R_0} u_0(x) \\ &= \|u_0\|_{L^1} - 2\int_{|x|\leqslant R_0} u_0(x) \\ &\leqslant -\tfrac{1}{2}\|u_0\|_{L^1} \leqslant -\frac{1}{2}\int_{|x|\leqslant R_0} u_0(x).\end{aligned}$$

Hence

$$\begin{aligned}\iint_K V(x)u^p\varphi &\leqslant Mc_0^{1-q}R^{-e} - \frac{q}{2}\int_{|x|\leqslant R_0} u_0(x) \\ &\leqslant Mc_0^{1-q}R^{-e} - 2Mc_0^{1-q}R^{-e} \\ &= -Mc_0^{1-p'}R^{-e} < 0,\end{aligned}$$

which is absurd because $V, u \geqslant 0$. Therefore there is no global weak solution to (1). □

Remark 2 Examples of u_0 satisfying the conditions of Theorem 2 are

$$u_0 = a\chi_{B_{R_0}(0)}$$

where $R_0 \geqslant C_0^{1/2}$ and a is a constant satisfying

$$a \geqslant \frac{4Mc_0^{1-q}}{\omega_n q} C_0^{-(n+e)/2}.$$

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