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Two new Levenberg-Marquardt methods for nonsmooth nonlinear complementarity problems

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ABSTRACT: The Levenberg-Marquardt method and its variants are of particular importance for solving nonsmooth systems of equations. In this paper, we present two kinds of new Levenberg-Marquardt method for nonsmooth nonlinear complementarity problems. Under some assumptions, the present methods are shown to be convergent. Results of numerical experiments are also given.

KEYWORDS: convergence

INTRODUCTION

Nonlinear complementarity problem have been proposed in the study of the nonlinear programming problems, the variational inequality, equilibrium problems, and engineering mechanics[1–3]. There has been intense research on nonlinear complementarity problems and related nonlinear equations[4,5]. In the past few years, there has been a growing interest in the study of nonsmooth nonlinear complementarity problems[6,7]. In this paper, we focus on the nonsmooth nonlinear complementarity problem

$$F(x) \geqslant 0, \quad Z(x) \geqslant 0, \quad Z(x)^{\mathrm{T}} F(x) = 0, \qquad (1)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian and $Z : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. When $Z(x) = x$, (1) is the problem, which has been considered in Refs. 2, 3.

In the next section, we recall some results of generalized Jacobian and semismoothness. We then give the new Levenberg-Marquardt methods for the nonsmooth nonlinear complementarity problem. The convergence results of the new Levenberg-Marquardt algorithms are also given. Finally, numerical experiments are described.

PRELIMINARIES

A quantity with a subscript k denotes that quantity evaluated at x_k. The vector norm is the l_2 norm. We write $F(x) = (f_1(x), \dots, f_n(x))^{\mathrm{T}}$, $Z(x) = (z_1(x), \dots, z_n(x))^{\mathrm{T}}$.

Let H be locally Lipschitzian. Then H is almost everywhere F-differentiable. Let the set of points where H is F-differentiable be denoted by D_H. Then for $x \in \mathbb{R}^n$,

$$\partial_{\mathrm{B}} H(x) = \{V \in \mathbb{R}^{n\times n} \mid \exists\{x_k\} \in D_H, \{x_k\} \to x, \{H'(x_k)\} \to V\}.$$

The general Jacobian of $H(x) : \mathbb{R}^n \to \mathbb{R}^n$ at x in the sense of Clark is

$$\partial H(x) = \operatorname{conv} \partial_{\mathrm{B}} H(x).$$

Proposition 1 (Ref. 3) *If $\partial_{\mathrm{B}} H(x)$ is a nonempty and compact set for any x, the point to set B-subdifferential map is upper semicontinuous.*

Definition 1 $H(x)$ is semismooth at x if $H(x)$ is locally Lipschitz at x and

$$\lim_{\substack{V \in \partial H(x+th') \\ h' \to h,\, t\downarrow 0}} Vh'$$

exists for any $h \in \mathbb{R}^n$. If $H(x)$ is semismooth at x, we know $Vh - H'(x;h) = o(\|h\|)$, $\forall V \in \partial H(x+h)$, $h \to 0$. If for any $V \in \partial H(x+h)$, $h \to 0$, $Vh - H'(x;h) = o(\|h\|^2)$, we say that the function $H(x)$ strongly semismooth at x.

Proposition 2 (Ref. 3) *If $H(x) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and semismooth at x, then*

$$\lim_{\substack{V \in \partial H(x+th) \\ h \to 0}} \frac{\|H(x+h) - H(x) - Vh\|}{\|h\|} = 0.$$

If $H(x) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, strongly semismooth at x, and directionally differentiable in a neighbourhood of x, then

$$\limsup_{\substack{V \in \partial H(x+th) \\ h \to 0}} \frac{\|H(x+h) - H(x) - Vh\|}{\|h\|^2} < \infty.$$

NEW LEVENBERG-MARQUARDT METHODS AND THEIR CONVERGENCE

Evidently, the above nonsmooth nonlinear complementarity problem can be reformulated as the nonsmooth equation

$$\min\{z_i(x), f_i(x)\} = 0, i = 1, \dots, n. \tag{2}$$

Let

$$G(x) = (g_1(x), \dots, g_n(x))^{\mathrm{T}},$$

where

$$g_i(x) = \min\{z_i(x), f_i(x)\}, \quad i = 1, \dots, n.$$

Hence (2) can be rewritten as

$$G(x) = 0. \tag{3}$$

In Ref. 7 they define the set-valued mapping $x \to V(x)$ as

$$V(x) = V_1(x) \times \cdots \times V_n(x), \tag{4}$$

where

$$V_i(x) = \begin{cases} \{\nabla z_i(x)\}, & z_i(x) \leqslant f_i(x); \\ \partial_{\mathrm{B}} f_i(x), & z_i(x) > f_i(x). \end{cases}$$

Newton's method for solving the nonsmooth nonlinear complementarity problem is given by

$$x_{k+1} = x_k - \xi_k^{-1} G(x_k), \quad \xi_k \in V(x_k).$$

Because $V(x)$ is not a subdifferential of $G(x)$ and not even upper-semicontinuous, a kind of subdifferential of $G(x)$ was also given in Ref. 7 by the set-valued mapping

$$\bar{V}(x) = \bar{V}_1(x) \times \cdots \times \bar{V}_n(x),$$

where

$$\bar{V}_i(x) = \begin{cases} \{\nabla z_i(x)\}, & z_i(x) < f_i(x); \\ \{\nabla z_i(x)\} \bigcup \partial_{\mathrm{B}} f_i(x), & z_i(x) = f_i(x); \\ \partial_{\mathrm{B}} f_i(x), & z_i(x) > f_i(x). \end{cases}$$

It is easy to see that $V(x) \subset \bar{V}(x)$ for $\forall x \in \mathbb{R}^n$. The following propositions give the properties of $\bar{V}(x)$.

Proposition 3 (Ref. 7) *The set-valued mapping $\bar{V}(x)$ is upper-semicontinuous.*

Proposition 4 (Ref. 7) *Suppose $x_0 \in \mathbb{R}^n$. If all $V_0 \in \bar{V}(x_0)$ are nonsingular, then there exists $\bar{\beta} > 0$ such that*

$$\|V_0^{-1}\| \leqslant \bar{\beta}, \quad \forall V_0 \in V(x_0).$$

The Levenberg-Marquardt method is one of the most used methods for solving optimization problems[8,9]. We are now in the position to consider the local versions of Levenberg-Marquardt type methods for the nonsmooth nonlinear complementarity problem (1). Similar methods have also been mentioned in Ref. 10. Given a starting vector $x_0 \in \mathbb{R}^n$, let $x_{k+1} = x_k + d_k$, where d_k is the solution of the system

$$((V_k)^{\mathrm{T}} V_k + \sigma_k I) d = -(V_k)^{\mathrm{T}} G(x_k), \\ V_k \in \bar{V}(x_k), \quad \sigma_k \geqslant 0.$$

In the inexact versions of this method, d_k can be given by the solution of the system

$$\begin{aligned} ((V_k)^{\mathrm{T}} V_k + \sigma_k I) d &= -(V_k)^{\mathrm{T}} G(x_k) + r_k, \\ V_k &\in \bar{V}(x_k), \quad \sigma_k \geqslant 0, \end{aligned} \tag{5}$$

where r_k is the vector of residuals and we can assume $\|r_k\| \leqslant \alpha_k \|(V_k)^{\mathrm{T}} G(x_k)\|$ for some $\alpha_k \geqslant 0$. We now give a local convergence Levenberg-Marquardt type method for (1).

Algorithm 1 Levenberg-Marquardt Method I

Step 1: We are given x_0, $\epsilon > 0$, $\lambda_i^k \in \mathbb{R}^n$, $0 < |\lambda_i^k| < +\infty$.

Step 2: Solve the system to get d_k:

$$\begin{aligned} &((V_k)^{\mathrm{T}} V_k + \operatorname{diag}(\lambda_i^{(k)} g_i(x_k))) d = \\ &\quad -(V_k)^{\mathrm{T}} G(x_k) + r_k, \quad V_k \in V(x_k), \end{aligned} \tag{6}$$

for $i = 1, \dots n$ and r_k is the vector of residuals

$$\|r_k\| \leqslant \alpha_k \|(V_k)^{\mathrm{T}} G(x_k)\|, \quad \alpha_k \geqslant 0.$$

Step 3: Set $x_{k+1} = x_k + d_k$. If $\|G(x_k)\| \leqslant \epsilon$, terminate. Otherwise, let $k := k + 1$ and go to Step 2.

Based upon the above analysis, we give the following local convergence result for Algorithm 1.

Lemma 1 *Suppose that $x^\star$ is a solution of the problem (1). We have*

$$\|\operatorname{diag}(\lambda_i^{(k)} g_i(x_k))\| \leqslant M,$$

for $\forall x \in U(x^\star, \delta)$, for $\lambda_i^{(k)} \in \mathbb{R}$, $i = 1, \dots, n$.

Proof: Use the fact that the function in (1) is locally Lipschitzian and continuous. □

Theorem 1 *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0$, $\alpha_k \leqslant a$ for all k. Let $x^\star$ be a solution of the problem (1), G be semismooth at $x^\star$, and all $V_\star \in \bar{V}(x_\star)$ be nonsingular. Then the sequence $\{x_k\}$ converge Q-linearly to $x^\star$ for $\|x_0 - x^\star\| \leqslant \epsilon$.*

Proof: By Lemma 1, for all x_k sufficiently close to $x^\star$, we get

$$\|((V_k)^{\mathrm{T}}V_k + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k)))^{-1}\| \leqslant \frac{\bar{\beta}^2}{1-\bar{\beta}^2(M+\epsilon)}.$$

Let $C = \bar{\beta}^2/(1-\bar{\beta}^2(M+\epsilon))$. We have

$$\|((V_k)^{\mathrm{T}}V_k + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k)))^{-1}\| \leqslant C.$$

Furthermore, by Proposition 2, there exists $\bar{\delta} > 0$, which can be taken arbitrarily small, such that

$$\|G(x_k) - G(x^\star) - V_k(x_k - x^\star)\| \leqslant \bar{\delta}\|x_k - x^\star\|$$

for all x_k in a sufficiently small neighbourhood of $x^\star$ depending on $\bar{\delta}$. By Proposition 3 and the upper semicontinuity of the $\bar{V}(x)$, we also know that for all $V_k \in V(x) \subset \bar{V}(x)$ and all x_k sufficiently close to $x^\star$

$$\|(V_k)^{\mathrm{T}}\| \leqslant c_1,$$

where $c_1 > 0$ is a suitable constant From the locally Lipschitz continuity of $G(x)$, we have

$$\begin{aligned}\|(V_k)^{\mathrm{T}}G(x_k)\| &\leqslant \|(V_k)^{\mathrm{T}}\|\|G(x_k) - G(x^\star)\| \\ &\leqslant c_1L\|x_k - x^\star\|,\end{aligned}$$

for all x_k in a sufficiently small neighbourhood of $x^\star$ and a constant $L > 0$. We also know that

$$\begin{aligned}&[(V_k)^{\mathrm{T}}V_k + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k))](x_{k+1} - x^\star) \\ &\quad= [(V_k)^{\mathrm{T}}V_k + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k))](x_k - x^\star) \\ &\qquad - (V_k)^{\mathrm{T}}G(x_k) + r_k \\ &\quad= (V_k)^{\mathrm{T}}[G(x^\star) - G(x_k) + V_k(x_k - x^\star)] \\ &\qquad + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k))(x_k - x^\star) + r_k.\end{aligned}$$

Multiplying the above equation by $[(V_k)^{\mathrm{T}}V_k + \operatorname{diag}(\lambda_i^{(k)}g_i(x_k))]^{-1}$ and taking into account the above results, we get

$$\begin{aligned}\|x_{k+1} - x^\star\| &\leqslant C(\|(V_k)^{\mathrm{T}}\|\|G(x_k) - G(x^\star) \\ &\qquad - V_k(x_k - x^\star)\| + \|\operatorname{diag}(\lambda_i^{(k)}g_i(x_k))\| \\ &\qquad \|x_k - x^\star\| + a\|(V_k)^{\mathrm{T}}G(x_k)\|) \\ &\leqslant C(c_1\delta\|x_k - x^\star\| \\ &\qquad + M\|x_k - x^\star\| + ac_1L\|x_k - x^\star\|) \\ &= C(c_1\bar{\delta} + M + ac_1L)\|x_k - x^\star\|.\end{aligned}$$

Let $\vartheta = C(c_1\bar{\delta} + M + ac_1L)$. Then

$$\|x_{k+1} - x^\star\| \leqslant \vartheta\|x_k - x^\star\|.$$

Since $\bar{\delta}$ can be chosen to be arbitrarily small, by taking x_k sufficiently close to $x^\star$, there exist $M > 0$ and $a > 0$ such that $\vartheta < 1$, so that the Q-linear convergence of $\{x_k\}$ to $x^\star$ follows by taking $\|x_0 - x^\star\| \leqslant \epsilon$ for a small enough $\epsilon > 0$. Thus we complete the proof of the theorem. □

Theorem 2 *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0$, $\alpha_k \leqslant a$ for all k such that $\|r_k\| \leqslant \alpha_k\|G(x_k)\|$, $\alpha_k \geqslant 0$. Let $x^\star$ be a solution of the problem (1), G be semismooth at $x^\star$, and all $V_\star \in \bar{V}(x_\star)$ be nonsingular. Then the sequence $\{x_k\}$ converge Q-linearly to $x^\star$ for $\|x_0 - x^\star\| \leqslant \epsilon$.*

The proof of Theorem 2 is similar to that for Theorem 1 so we omit it.

Remark 1 Theorem 1 and Theorem 2 hold with $\|r_k\| = 0$.

Remark 2 In Algorithm 1, if d_k is computed by (5) instead of (6), then Theorem 1, Theorem 2, and Remark 1 can also be obtained.

In the following, we give the global versions of the Levenberg-Marquardt type method for the above nonsmooth nonlinear complementarity problem (1). In this section, we assume that the merit function

$$\Psi(x) = \tfrac{1}{2}\|G(x)\|^2$$

is continuously differentiable.

We now describe the new Levenberg-Marquardt method for (1). The global convergence of the method is also given.

Algorithm 2 Levenberg-Marquardt Method II

Step 1: We are given $x_0 \in \mathbb{R}^n$, $\alpha, \beta, \gamma \in (0,1)$, $\mu_0 = \|G(x_0)\|^2$, $k := 0$.

Step 2: If $\|\nabla\Psi\| = 0$ then stop. Otherwise, compute d_k by the system of linear equations

$$(V_k^{\mathrm{T}} V_k + \mu_k I)d = -V_k^{\mathrm{T}} G(x_k), \qquad (7)$$

where $V_k \in V(x_k)$.

Step 3: If d_k satisfies

$$\|G(x_k + d_k)\| \leqslant \gamma \|G(x_k)\|, \qquad (8)$$

set $x_{k+1} = x_k + d_k$, otherwise compute x_{k+1} by $x_{k+1} = x_k + \beta^{m_k} d_k$, where m_k is the smallest nonnegative integer m such that

$$\Psi(x_k+\beta^m d_k) - \Psi(x_k) \leqslant \alpha\beta^m \nabla\Psi(x_k)^{\mathrm{T}} d_k. \qquad (9)$$

If $\|\nabla\Psi\| = 0$, terminate. Otherwise, let $\mu_{k+1} = \|G(x_{k+1})\|^2$, $k := k + 1$, and go to Step 2.

In the following, we give the global convergence result.

Theorem 3 *$\{x_k\}$ is a sequence generated by Algorithm 2. Then any accumulation point x^* of $\{x_k\}$ is a stationary points of Ψ.*

Proof: Since $\nabla\Psi \neq 0$ implies $d_k \neq 0$, from (7) and the above analysis, we get

$$\nabla\Psi(x_k)^{\mathrm{T}} d_k = (V_k^{\mathrm{T}} V_k)^{\mathrm{T}} d_k = -((V_k^{\mathrm{T}} V_k + \mu_k I)d_k)^{\mathrm{T}} d_k < 0.$$

By (8) and (9), we know $\{\Psi(x_k)\}$ is monotonically decreasing and μ_k has a limit. If $\mu_k \to 0$, then any accumulation point of $\{x_k\}$ is a solution of (1). If

$$\lim_{k\to\infty} \mu_k = \bar{\mu} > 0,$$

then we have

$$\nabla\Psi(x_k)^{\mathrm{T}} d_k = -((V_k^{\mathrm{T}} V_k + \mu_k I)d_k)^{\mathrm{T}} d_k < -\bar{\mu}\|d_k\|^2.$$

Hence any accumulation point of $\{x_k\}$ is a solution of (1). So we complete the proof. □

Remark 3 The different merit functions can be based on the well-known Fischer-Burmeister function

$$\varphi(a, b) = \sqrt{a^2 + b^2} - a - b.$$

Then we can see that (1) is equivalent to the problem

$$\Phi(x) = \begin{pmatrix} \psi(F_1, Z_1) \\ \vdots \\ \psi(F_n, Z_n) \end{pmatrix} = 0.$$

Table 1 Numerical tests for Example 1 using Algorithm 1.

iteration	x_k	$G(x)$
2	$(0.5009, 1.0000)^{\mathrm{T}}$	$(0.2509, 0.1254)^{\mathrm{T}}$
3	$(0.2509, 1.0000)^{\mathrm{T}}$	$(0.0630, 0.0315)^{\mathrm{T}}$
4	$(0.1257, 1.0000)^{\mathrm{T}}$	$(0.0158, 0.0079)^{\mathrm{T}}$
5	$(0.0630, 1.0000)^{\mathrm{T}}$	$(0.0039, 0.0019)^{\mathrm{T}}$
6	$(0.0315, 1.0000)^{\mathrm{T}}$	$(0.99, 0.49)^{\mathrm{T}} \times 10^{-3}$
7	$(0.0158, 1.0000)^{\mathrm{T}}$	$(0.25, 0.12)^{\mathrm{T}} \times 10^{-3}$
8	$(0.0079, 1.0000)^{\mathrm{T}}$	$(0.62, 0.31)^{\mathrm{T}} \times 10^{-4}$
9	$(0.0039, 1.0000)^{\mathrm{T}}$	$(0.15, 0.07)^{\mathrm{T}} \times 10^{-4}$
10	$(0.0019, 1.0000)^{\mathrm{T}}$	$(0.39, 0.19)^{\mathrm{T}} \times 10^{-5}$
11	$(0.0009, 1.0000)^{\mathrm{T}}$	$(0.99, 0.49)^{\mathrm{T}} \times 10^{-6}$
12	$(0.0004, 1.0000)^{\mathrm{T}}$	$(0.24, 0.12)^{\mathrm{T}} \times 10^{-6}$
13	$(0.0002, 1.0000)^{\mathrm{T}}$	$(0.62, 0.31)^{\mathrm{T}} \times 10^{-7}$
14	$(0.0001, 1.0000)^{\mathrm{T}}$	$(0.15, 0.07)^{\mathrm{T}} \times 10^{-7}$
15	$(0.0000, 1.0000)^{\mathrm{T}}$	$(0.39, 0.19)^{\mathrm{T}} \times 10^{-8}$

The merit function $\Psi(x)$ in the above methods can be defined by

$$\Psi(x) = \tfrac{1}{2}\|\Phi(x)\|.$$

Then the above merit function is a continuously differentiable map and the Fischer-Burmeister function can also be replaced by the family of new nonlinear complementarity problem functions [11]

$$\phi_{\mathrm{new}}(a, b) = \|(a, b)\|_p - (a + b),$$

where p is any fixed real number in the interval $(1, +\infty)$ and $\|(a, b)\|_p$ denotes the p-norm of (a, b). Then we can see that (1) is equivalent to the problem

$$\Phi_p(x) = \begin{pmatrix} \psi_{\mathrm{new}}(F_1, Z_1) \\ \vdots \\ \psi_{\mathrm{new}}(F_n, Z_n) \end{pmatrix} = 0.$$

The merit function $\Psi(x)$ in the above methods can also be defined by

$$\Psi_p(x) = \tfrac{1}{2}\|\Phi_p(x)\|.$$

By some assumptions, we can give the global convergence results of Algorithm 2 by using $\Psi_p(x)$.

NUMERICAL TESTS

In this section, in order to show the performance of the above new Levenberg-Marquardt type methods, we present some numerical results for them. The results indicate that the new Levenberg-Marquardt algorithms work quite well in practice.

Example 1 We consider the complementarity problem

$$F(x) \geqslant 0, Z(x) \geqslant 0, \quad Z(x)^{\mathrm{T}} F(x) = 0,$$

Table 2 Numerical tests for Example 1 using Algorithm 2.

iteration	x_k	$G(x)$
2	$(0.5009, 1.0000)^{\mathrm{T}}$	$(0.2509, 0.1254)^{\mathrm{T}}$
3	$(0.1866, 1.0000)^{\mathrm{T}}$	$(0.0348, 0.0174)^{\mathrm{T}}$
4	$(0.1703, 1.0000)^{\mathrm{T}}$	$(0.0290, 0.0145)^{\mathrm{T}}$
5	$(0.1580, 1.0000)^{\mathrm{T}}$	$(0.0249, 0.0124)^{\mathrm{T}}$
6	$(0.1481, 1.0000)^{\mathrm{T}}$	$(0.0219, 0.0109)^{\mathrm{T}}$
7	$(0.1400, 1.0000)^{\mathrm{T}}$	$(0.0196, 0.0098)^{\mathrm{T}}$
8	$(0.1331, 1.0000)^{\mathrm{T}}$	$(0.0177, 0.0088)^{\mathrm{T}}$
9	$(0.1272, 1.0000)^{\mathrm{T}}$	$(0.0161, 0.0080)^{\mathrm{T}}$
10	$(0.1221, 1.0000)^{\mathrm{T}}$	$(0.0149, 0.0074)^{\mathrm{T}}$
11	$(0.1175, 1.0000)^{\mathrm{T}}$	$(0.0138, 0.0069)^{\mathrm{T}}$
12	$(0.1134, 1.0000)^{\mathrm{T}}$	$(0.0128, 0.0064)^{\mathrm{T}}$
13	$(0.1098, 1.0000)^{\mathrm{T}}$	$(0.0120, 0.0060)^{\mathrm{T}}$

where the functions

$$F(x_1, x_2) = (\max\{x_1^2, \tfrac{5}{6}x_1^2\}, \max\{x_1^2, x_1^2 + 3x_2^2\})^{\mathrm{T}},$$
$$Z(x_1, x_2) = (2x_1^2 + x_2^2 + 10, \tfrac{1}{2}x_1^2)^{\mathrm{T}}.$$

We use Algorithm 1 to compute Example 1. Results for Example 1 with initial point $x_0 = (0.1, 0.7)^{\mathrm{T}}$, $\alpha_k \equiv 1$ and $\lambda_1 = 0.01$, $\lambda_2 = 0.01$, $\epsilon = 1 \times 10^{-4}$ are presented in Table 1.

Now we use Algorithm 2 to compute Example 1. Results for Example 1 with initial point $x_0 = (1, 1)^{\mathrm{T}}$ and $\lambda_1 = 0.01$, $\lambda_2 = 1$, $\epsilon = 1 \times 10^{-4}$ are presented in Table 2.

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