RESEARCH ARTICLE

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Confidence intervals for multivariate value at risk

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ABSTRACT: Confidence intervals for the γ-quantile of a linear combination of N non-normal variates with a linear dependence structure would be useful to the financial institutions as the intervals enable the accuracy of the value at risk (VaR) of a portfolio of investments to be quantified. Here we construct $100(1-\alpha)\%$ confidence intervals for the γ-quantile using procedures based on bootstrap, normal approximation and hypothesis testing. We show that the method based on hypothesis testing produces a confidence interval which is more satisfactory than those found by using bootstrap or normal approximation.

KEYWORDS: non-normal variates, γ-quantile, bootstrap, hypothesis testing

INTRODUCTION

Consider a portfolio consisting of N stocks. The absolute value of the γ-quantile of the return of the portfolio is called the value at risk (VaR) of the portfolio.

VaR has been frequently used by commercial and investment banks to capture the potential loss in value of their traded portfolios from adverse market movements over a specified period.

To evaluate VaR in the multivariate situation where N stocks are involved, we usually begin with the evaluation of a multivariate distribution for the N stocks. A common approach is to fit the data on returns by the multivariate version of the normal, Student t or skewed Student t distribution. Other approaches may take into account the tail-dependence[1], and asymmetry[2–5]. A more sophisticated approach is one which is based on copulas[2,6–13].

Presently we use an approach based on a type of non-normal distribution called the quadratic-normal distribution[14,15]. To describe the approach, we first let $\mathbf{S} = (S_1, S_2, \dots, S_N)^{\mathrm{T}}$ be a vector of uncorrelated variates of which S_i can be expressed as

$$S_i = \begin{cases} \lambda_{i1} e_i + \lambda_{i2}(e_i^2 - \frac{1}{2}[1+\lambda_{i3}]), & e_i \geqslant 0, \\ \lambda_{i1} e_i + \lambda_{i2}(\lambda_{i3} e_i^2 - \frac{1}{2}[1+\lambda_{i3}]), & e_i < 0 \end{cases}$$

where $e_i \sim N(0,1)$. The variate S_i is said to have a quadratic-normal distribution with parameters 0 and $\boldsymbol{\lambda}_i$, and we may write $S_i \sim QN(0, \boldsymbol{\lambda}_i)$. The mean of S_i is 0 while the kth moment of S_i is given by $m_k = E(S_i^k)$, $k = 2, 3, 4$. The standardized moments

$$\bar{m}_3 = m_3/m_2^{3/2}, \quad \bar{m}_4 = m_4/m_2^2$$

will then be, respectively, the measures of skewness and kurtosis of S_i. Next, let $\mathbf{A}$ be an $N \times N$ orthogonal matrix, $\boldsymbol{\mu}$ an $N \times 1$ vector of constants and $\mathbf{R} = (R_1, R_2, \dots, R_N)^{\mathrm{T}}$ an $N \times 1$ vector given by

$$\mathbf{R} = \boldsymbol{\mu} + \mathbf{AS}, \tag{1}$$

$\mathbf{w} = (w_1, w_2, \dots, w_N)^{\mathrm{T}}$ a vector of constants with $\sum_{i=1}^N w_i = 1$ and

$$R = \sum_{i=1}^N w_i R_i. \tag{2}$$

When $\lambda_{i3} = -1$ and λ_{i2} is large, the distribution of the random variable S_i will have fat tails and narrow waist. As the matrix $\mathbf{A}$ represents an orthogonal transformation, and the vector $\boldsymbol{\mu}$, on the other hand represents a translation, the distribution of R_i will also have fat tails and narrow waist. As the distribution of stock return often also has fat tails and narrow waist, and the returns of different stocks are usually correlated, the distribution of $\mathbf{R}$ given by (1) can be used to model the joint distribution of the returns of N stocks. For a portfolio of N stocks, the portfolio return can be represented by R given by (2).

Let F_R be the cumulative distribution function of R and assume that the γ-quantile, $Q_\gamma = F_R^{-1}(\gamma)$, is uniquely defined. When γ is small, the absolute value of Q_γ will represent the VaR which has a confidence level of $100(1-\gamma)\%$.

After finding an estimate for the VaR, it is usually desirable to access the accuracy of the VaR estimate by constructing a confidence interval for the VaR.

The layout of the paper is as follows. In the next three sections, we describe, respectively, the procedures based on bootstrap, normal approximation and hypothesis testing for finding a confidence interval for the VaR. We then compare the performance of the three methods for constructing confidence intervals for the VaR. In the last section, we give an example which shows that multivariate quadratic-normal distribution is able to fit a real data set obtained from the Kuala Lumpur stock exchange.

BOOTSTRAP CONFIDENCE INTERVAL FOR γ-QUANTILE

First, let $(r_{1j}, r_{2j}, \dots, r_{Nj})$ be the jth observed value of $\mathbf{R}$, $j = 1, 2, \dots, n$. From the n observed values $(r_{1j}, r_{2j}, \dots, r_{Nj}), j = 1, 2, \dots, n$, we first compute the (k, l) entry of the matrix $\hat{\mathbf{V}}$ of the estimated variance-covariance of $\mathbf{R}$ as shown below: $\hat{v}_{kl} = (1/n)\sum_{j=1}^{n} r_{kj}r_{lj} - \hat{\mu}_k\hat{\mu}_l$ where $\hat{\mu}_k = (1/n)\sum_{j=1}^{n} r_{kj}$.

We next compute $\hat{\mathbf{A}} = [\hat{\mathbf{a}}_1\hat{\mathbf{a}}_2\cdots\hat{\mathbf{a}}_N]$ where $\hat{\mathbf{a}}_i$ is the ith eigenvector of $\hat{\mathbf{V}}$, and $\|\hat{\mathbf{a}}_i\| = 1$. By using $\hat{\mathbf{A}}$, we compute

$$\begin{pmatrix} s_{1j} \\ s_{2j} \\ \vdots \\ s_{Nj} \end{pmatrix} = \hat{\mathbf{A}}^{\mathrm{T}} \begin{pmatrix} r_{1j} - \hat{\mu}_1 \\ r_{2j} - \hat{\mu}_2 \\ \vdots \\ r_{Nj} - \hat{\mu}_N \end{pmatrix}, \; j = 1, 2, \dots, n.$$

By using the constrained maximum likelihood procedure[16], we find the quadratic-normal distributions $QN(0, \hat{\boldsymbol{\lambda}}_i)$ and $QN(\hat{\mu}, \hat{\boldsymbol{\lambda}})$ which fit $s_{i1}, s_{i2}, \dots, s_{in}$ and the n observed values of R. Let z_γ be the $(1-\gamma)$-quantile of the standard normal distribution. An estimate of the γ-quantile of R is then given by

$$\hat{Q} = \hat{\mu} + \hat{\lambda}_1(-z_\gamma) + \hat{\lambda}_2\left[\hat{\lambda}_3(-z_\gamma)^2 - \frac{1+\hat{\lambda}_3}{2}\right].$$

Next we generate B values of $(\tilde{r}_{1j}, \tilde{r}_{2j}, \dots, \tilde{r}_{Nj})$, $(j = 1, 2, \dots, n)$, using

$$\begin{pmatrix} \tilde{r}_{1j} \\ \tilde{r}_{2j} \\ \vdots \\ \tilde{r}_{Nj} \end{pmatrix} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_N \end{pmatrix} + \hat{\mathbf{A}} \begin{pmatrix} \tilde{s}_{1j} \\ \tilde{s}_{2j} \\ \vdots \\ \tilde{s}_{Nj} \end{pmatrix}$$

where $\tilde{s}_{ij} \sim QN(0, \hat{\boldsymbol{\lambda}}_i), j = 1, 2, \dots, n; i = 1, 2, \dots, N$.

By using the constrained maximum likelihood procedure, we find the quadratic-normal distribution $QN(\tilde{\mu}, \tilde{\boldsymbol{\lambda}})$ which fits the values $\tilde{r}_j = \sum_{i=1}^{N} w_i\tilde{r}_{ij}, j = 1, 2, \dots, n$. Next let

$$\tilde{Q} = \tilde{\mu} + \tilde{\lambda}_1(-z_\gamma) + \tilde{\lambda}_2\left[\tilde{\lambda}_3(-z_\gamma)^2 - \frac{1+\tilde{\lambda}_3}{2}\right]$$

be the estimated quantile, and $QN(\tilde{\mu}^*, \tilde{\boldsymbol{\lambda}}^*)$ the quadratic-normal distribution which fits the B values of $\tilde{Q}$.

The approximately-$100(1-\alpha)\%$ bootstrap confidence interval for the γ-quantile is then given by $[Q_{\mathrm{L}}, Q_{\mathrm{U}}]$ where

$$Q_{\mathrm{L}} = \tilde{\mu}^* + \tilde{\lambda}_1^*(-z_{\alpha/2}) + \tilde{\lambda}_2^*\left[\tilde{\lambda}_3^*(-z_{\alpha/2})^2 - \frac{1+\tilde{\lambda}_3^*}{2}\right]$$

and

$$Q_{\mathrm{U}} = \tilde{\mu}^* + \tilde{\lambda}_1^*(z_{\alpha/2}) + \tilde{\lambda}_2^*\left[(z_{\alpha/2})^2 - \frac{1+\tilde{\lambda}_3^*}{2}\right].$$

CONFIDENCE INTERVALS BASED ON NORMAL APPROXIMATION

From the B values $\tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \dots, \tilde{Q}^{(B)}$ of $\tilde{Q}$ we can find the estimated variance

$$\tilde{\sigma}^2 = \frac{1}{B-1}\sum_{b=1}^{B}(\tilde{Q}^{(b)} - \bar{\tilde{Q}})^2$$

where $\bar{\tilde{Q}} = (1/B)\sum_{b=1}^{B}\tilde{Q}^{(b)}$. Then the approximately-$100(1-\alpha)\%$ confidence interval based on normal approximation for the γ-quantile is

$$[\hat{Q} - z_{\alpha/2}\tilde{\sigma}, \hat{Q} + z_{\alpha/2}\tilde{\sigma}].$$

PROCEDURE BASED ON HYPOTHESIS TESTING

Consider the problem of testing $H_0 : Q_\gamma = Q_\gamma^0$ against $H_1 : Q_\gamma \neq Q_\gamma^0$. Suppose we test the above H_0 by using the decision rule of accepting H_0 at the α level if $Q_{\mathrm{L}}^{(0)} \leqslant \hat{Q} \leqslant Q_{\mathrm{U}}^{(0)}$ where $Q_{\mathrm{L}}^{(0)}$ and $Q_{\mathrm{U}}^{(0)}$

are, respectively, the $100(\alpha/2)\%$ and $100(1-\alpha/2)\%$ points of the quadratic-normal distribution which is used to fit the B values of $\tilde{Q}$ obtained when the B values of $((\tilde{r}_{1j}, \tilde{r}_{2j}, \dots, \tilde{r}_{Nj}), j = 1, 2, \dots, n)$ are generated using

$$\begin{pmatrix} \tilde{r}_{1j} \\ \tilde{r}_{2j} \\ \vdots \\ \tilde{r}_{Nj} \end{pmatrix} = \begin{pmatrix} \mu_1^{(m)} \\ \mu_2^{(m)} \\ \vdots \\ \mu_N^{(m)} \end{pmatrix} + \mathbf{A}^{(m)} \begin{pmatrix} \tilde{s}_{1j} \\ \tilde{s}_{2j} \\ \vdots \\ \tilde{s}_{Nj} \end{pmatrix}$$

where $\tilde{s}_{ij} \sim QN(0, \boldsymbol{\lambda}_i^{(m)})$, $j = 1, 2, \dots, n$ and $(((\mu_i^{(m)}, \boldsymbol{\lambda}_i^{(m)}), i = 1, 2, \dots, N), \mathbf{A}^{(m)})$ is found as follows.

Firstly, for a given value of $(((\mu_i, \boldsymbol{\lambda}_i), i = 1, 2, \dots, N), \mathbf{A})$, we find the moment $m_k = E(R - E(R))^k$, $k = 2, 3, 4$. Let $(\mu, \boldsymbol{\lambda})$ be such that $\mu = E(R)$ and the kth central moment of the quadratic-normal distribution $QN(\mu, \boldsymbol{\lambda})$ is equal to m_k, $k = 2, 3, 4$. Then R is approximately distributed as $QN(\mu, \boldsymbol{\lambda})$. Finally, $(((\mu_i^{(m)}, \boldsymbol{\lambda}_i^{(m)}), i = 1, 2, \dots, N), \mathbf{A}^{(m)})$ is $(((\mu_i, \boldsymbol{\lambda}_i), i = 1, 2, \dots, N), \mathbf{A})$ which minimizes

$$D^2 = (\mu - \hat{\mu})^2 + (\lambda_1 - \hat{\lambda}_1)^2 + (\lambda_2 - \hat{\lambda}_2)^2 + (\lambda_2\lambda_3 - \hat{\lambda}_2\hat{\lambda}_3)^2$$

subject to

$$\mu + \lambda_1(-z_\gamma) + \lambda_2\left[\lambda_3(-z_\gamma)^2 - \frac{1+\lambda_3}{2}\right] = Q_\gamma^0.$$

An approximately-$100(1-\alpha)\%$ confidence interval for the γ-quantile of R is now given by $\{Q_\gamma^0$: The null hypothesis that $Q_\gamma = Q_\gamma^0$ is accepted at the α level$\}$.

NUMERICAL EXAMPLES

Fig. 1 shows 100 simulated bootstrap confidence intervals for the γ-quantile of R when $n = 50$, $\mu_1 = 0$, $\boldsymbol{\lambda}_1^{\mathrm{T}} = (0.32, 0.68, 0.065)$, $\mu_2 = 0$, $\boldsymbol{\lambda}_2^{\mathrm{T}} = (0.378, 0.639, 0.073)$,

$$\mathbf{A} = \begin{pmatrix} 0.3090 & 0.9511 \\ -0.9511 & 0.3090 \end{pmatrix}.$$

The upper limits of the 100 confidence intervals have been arranged in ascending order.

Figs. 2 and 3 show 100 possible confidence intervals based on normal approximation and hypothesis testing. As in Fig. 1, the upper limits of the 100 confidence intervals have been arranged in ascending order.

Figs. 1–3 show that the estimated coverage probability of the confidence interval based on hypothesis testing is closer to the target value 0.95 than those of the bootstrap confidence interval and the confidence interval based on normal approximation.

Further comparison of the 3 types of confidence

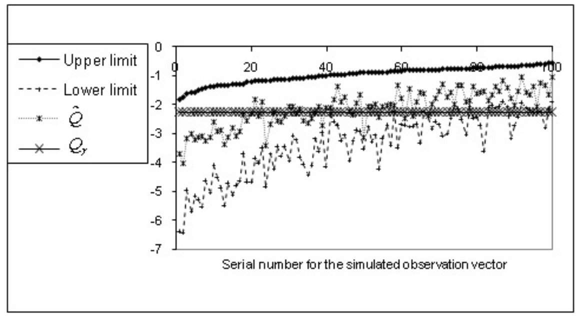

Fig. 1 100 simulated bootstrap confidence intervals for γ-quantile when $\gamma = 0.01$, $\alpha = 0.05$, $n = 50$, $B = 100$. Estimated coverage probability: 0.82; average length: 2.395, $\hat{Q}$: estimate of γ-quantile; Q_γ: true value of γ-quantile.

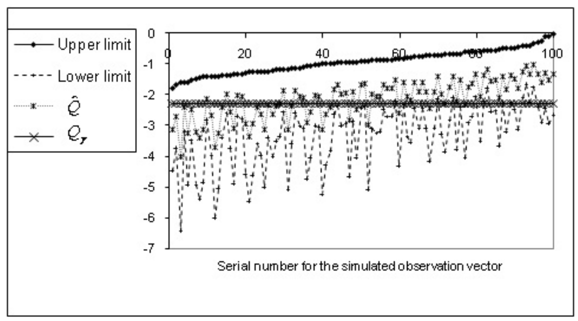

Fig. 2 100 simulated confidence intervals based on normal approximation for γ-quantile. Estimated coverage probability: 0.89, average length: 2.4296.

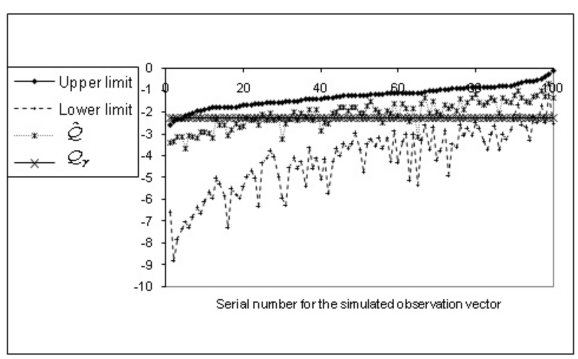

Fig. 3 100 simulated confidence intervals based on hypothesis testing procedure for γ-quantile. Estimated coverage probability: 0.91, average length: 2.9261.

Table 1 Estimated coverage probabilities and average lengths of confidence intervals for γ-quantile ($\gamma = 0.01$, $\alpha = 0.05$, $\mu_1 = \mu_2 = 0, n = 50$). $N = B = 100$, standard error of the estimated coverage probability $\approx$ 0.0218.

No	BTP	NAP	HTP	BTL	NAL	HTL
1	0.82	0.89	0.91	2.39207	2.42961	2.918
2	0.8	0.8	0.88	2.45177	2.49157	3.019
3	0.79	0.82	0.93	2.31436	2.35217	2.9195
4	0.72	0.72	0.83	2.17881	2.21492	2.634
5	0.91	0.89	0.94	0.76893	0.76353	1.214
6	0.85	0.86	0.91	1.26915	1.29416	1.8135
7	0.79	0.85	0.87	1.81021	1.83762	2.248
8	0.9	0.9	0.93	0.81293	0.8142	1.3455
9	0.71	0.69	0.82	2.16665	2.20639	2.5805
10	0.77	0.75	0.85	1.22067	1.24205	1.7185

BTP=Estimated coverage probability of confidence interval based on bootstrap.
NAP=Estimated coverage probability of confidence interval based on normal approximation.
HTP=Estimated coverage probability of confidence interval based on hypothesis testing.
BTL=Average length of confidence interval based on bootstrap.
NAL=Average length of confidence interval based on normal approximation.
HTL=Average length of confidence interval based on hypothesis testing.

Table 2 The parameters and measures of skewness and kurtosis of the quadratic-normal distribution.

No	$\bar{m}_3$	$\bar{m}_4$	$\boldsymbol{\lambda}_1^{\mathrm{T}}$ / $\boldsymbol{\lambda}_2^{\mathrm{T}}$		
1	3	16.6	0.322184	0.680924	0.065316
	2.8	15	0.377794	0.638861	0.072964
2	3.4	20.2	0.190061	0.770776	0.02769
	2.6	13.4	0.450006	0.589852	0.115554
3	3	16.7	0.300017	0.688155	0.020679
	2.4	12	0.502252	0.547528	0.125795
4	3.2	18.4	0.247697	0.72906	0.029551
	1	4.4	0.955589	0.177217	0.88736
5	−0.2	2.4	1.20867	−0.17517	−0.52415
	2	9.4	0.62038	0.454663	0.189261
6	0.6	3.1913	1.0745	0.055572	2.958
	1.6	7.2	0.732179	0.359491	0.270896
7	2	9.3	0.603592	0.45809	0.151303
	0.4	2.8	1.11827	−0.00217	−71.0956
8	−0.4	2.8	1.11826	−0.15408	−0.01403
	0.8	3.8	0.99794	0.126695	1.16716
9	3.6	22.6	0.054687	0.838405	−0.08815
	0.4	2.7	1.15398	−0.02222	−8.11265
10	−1	10.6	0.171503	0.387604	−1.47341
	1.4	6.2	0.797129	0.305546	0.361719

intervals can be found in Table 1 which displays the estimated coverage probabilities and average lengths for 10 values of $(\mu_1, \boldsymbol{\lambda}_1, \mu_2, \boldsymbol{\lambda}_2, \mathbf{A})$. The 10 values of $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are displayed in Table 2. The measures of skewness and kurtosis ($\bar{m}_3$ and $\bar{m}_4$) of the quadratic-normal distribution with the given $\boldsymbol{\lambda}_i$ are also included in Table 2. Table 1 shows that the coverage probability of the confidence interval based on hypothesis testing is closer to the target value 0.95 than those of the bootstrap confidence interval and confidence interval based on normal approximation.

Table 1 also shows that the average length of the confidence interval based on the hypothesis testing is longer than those of the bootstrap confidence interval and confidence interval based on normal approximation. This is not surprising because in order to have a larger coverage probability, the length of the confidence interval should be made longer.

APPLICATIONS IN FINANCE

The random variables $R_1, R_2, \dots, R_N$ in the first section may be considered to be the returns of N stocks, and the γ-quantile Q_γ of R becomes the value at risk (VaR) of the portfolio consisting of these N stocks. Thus if we can show that $\mathbf{R}$ can be written as $\mathbf{R} = \boldsymbol{\mu} + \mathbf{AS}$ of which $S_1, S_2, \dots, S_N$ are uncorrelated and $S_i \sim QN(0, \boldsymbol{\lambda}_i)$, then the methods in the second and fourth sections can be applied to find confidence intervals for the VaR of the portfolio.

In the following analysis, the data obtained from the Kuala Lumpur Stock Exchange (KLSE) are used. The data are the daily stock prices of three companies, namely Genting Bhd., Gamuda Bhd. and Tanjong PLC Bhd. in the KLSE from Thomson Financial Datastream (01/01/1993 to 31/8/2002). The data for the period from 01/07/1997 to 30/06/1999 are excluded in the present investigation because these data were collected during the financial crisis in SE Asia. The following results in the forms of table and figure are extracted from Ref. 15.

The variance-covariance matrix associated with the portfolio is

$$\begin{pmatrix} 4.6316 & 0.7453 & 1.2520 \\ 0.7453 & 4.0142 & 1.2299 \\ 1.2520 & 1.2299 & 5.7027 \end{pmatrix}.$$

Fig. 4 shows that the distribution of the portfolio returns R^{P} can be approximated well using the quadratic-normal distribution. Thus the methods in the second and fourth sections may be used to find confidence intervals for the VaR of the portfolio.

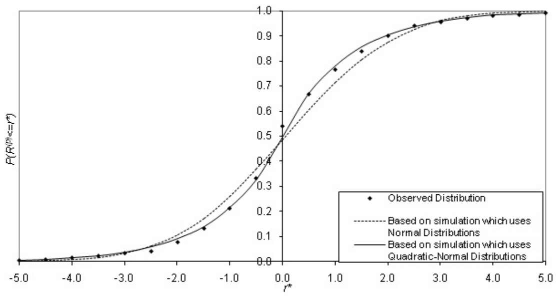

Fig. 4 Cumulative distribution of return for the portfolio.

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