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Exact solutions to some models of distributed-order time fractional diffusion equations via the Fox H functions

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ABSTRACT: In this article, three types of time-fractional diffusion equation of distributed order are introduced and some aspects of these equations are discussed. Using the appropriate joint integral transforms, fundamental solutions of these equations are obtained through the Fox H functions. The Mellin transform is an approach to change the fundamental solutions into the Fox H functions.

KEYWORDS: integral transform, partial fractional differential equation of distributed order

INTRODUCTION

The fractional differential operator of distributed order

$$D_{\mathrm{do}}^{\alpha}=\int_{l}^{u} b(\alpha)\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\,\mathrm{d}\alpha,\quad u>l\geqslant 0,\ b(\alpha)\geqslant 0,\quad (1)$$

is a generalization of the single order $D_{\mathrm{so}}^{\alpha}=\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ which by considering a continuous or discrete distribution of fractional derivative is obtained. The idea of fractional derivative of distributed order was stated by Caputo [1] and was developed by Caputo himself [2,3] and Bagley and Torvik [4,5] later. Other researchers used this idea. For example, Diethelm and Ford [6] used a numerical technique along with its error analysis to solve a distributed-order differential equation in engineering problems. Mainardi [7–10], Chechkin et al [11–13], Umarov et al [14], Kochubei [15], and Sun et al [16] investigated some linear distributed-order boundary value problems

$$\int_{0}^{m} b(\alpha)D^{\alpha}u(x,t)\,\mathrm{d}\alpha=B(D)u(x,t),$$
$$D=\frac{\mathrm{d}}{\mathrm{d}x},\qquad t>0,\ x\in\mathbb{R},$$

with pseudo-differential operator $B(D)$ and the Cauchy conditions

$$\frac{\partial^{k}}{\partial t^{k}}u(x,0^{+})=f_{k}(x)\quad k=0,1,\dots,m-1,\quad (2)$$

to treat the sub, normal and super diffusions as particular cases of the time-fractional diffusion equation of distributed order. These types of diffusion equations are related to the growth of the second moment (variance) denoted by $\sigma^2(t)$. It is a measure for the spatial spread of $u(x,t)$ with time of a random walking particle starting at the origin $x=0$ and initial condition $u(x,0)=\delta(x)$. On the nonlinear distributed-order differential equations applied in the models of viscoelasticity and system identification theory, Atanackovic [17,18] studied the existence and uniqueness of solution for equation

$$y''(t)+\int_{0}^{2} b(\alpha)D^{\alpha}y(t)\,\mathrm{d}\alpha=f(y,t),$$

where the function y is an unknown function. Now, in this paper in the distributed-order equations class we introduce three types of the time-fractional diffusion equations of distributed order (Klein-Gordon, Fokker-Planck and Giona-Roman equations of distributed order) and focus on mathematical aspects and technical approaches to find the explicit solutions of these equations. We find the fundamental solutions via the Fox H functions which is recalled after this section. We introduce the time-fractional Klein-Gordon equation of distributed order and using the joint Laplace-Fourier transform, we obtain the explicit solution via the Fox H functions. The Mellin transform is an alternative tool to change the solution into the Mellin-Barnes integral and construction of the Fox H functions. We proceed to study the time-fractional Fokker-Planck equation of distributed order and using the joint Laplace-Hankel transform its explicit solution is found via the Fox H functions. As a generalization of Giona-Roman equation, distributed-order case of this equation is written and by using the joint Laplace - $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform, ($\beta \geqslant 0$), the fundamental solution of this equation is given in the Laplace type integral of Fox H functions. This transform is a special case of the $\mathcal{L}_A$ transform introduced by Aghili and Ansari[19,20]

$$\mathcal{L}_A\{f(x);s\} = \int_0^\infty A'(x)\,\mathrm{e}^{-\Phi(s)A(x)} f(x)\,\mathrm{d}x,$$

for the increasing functions A and Φ. By writing some operational theorems for this transform authors used this transform for solving some partial fractional differential equations with non constant coefficients. Finally, the main conclusions are drawn.

FOX *H* FUNCTION

The Fox H function is a generalized hypergeometric function defined by means of a Mellin-Barnes type contour integral:

$$\begin{aligned} H^{m,n}_{p,q}(z) &= H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix}\right] \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s) z^s\,\mathrm{d}s, \qquad z \neq 0, \end{aligned}$$

where the integrand $\mathcal{H}$ is defined in terms of the Gamma functions

$$\begin{aligned} \mathcal{H}^{m,n}_{p,q}(s) &= \frac{A(s)B(s)}{C(s)D(s)}, \\ A(s) &= \prod_{k=1}^{m} \Gamma(b_k - B_k s), \\ B(s) &= \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s), \\ C(s) &= \prod_{k=m+1}^{q} \Gamma(1 - b_k + B_k s), \\ D(s) &= \prod_{j=n+1}^{p} \Gamma(a_j - A_j s), \end{aligned}$$

and the orders (m, n, p, q) are non-negative integers such that $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$. The parameters $A_j > 0$ and $B_k > 0$ are positive and a_j and b_k can be arbitrary complex such that the poles of the Gamma function entering the expressions $A(s)$ and $B(s)$ are simple poles and do not coincide: $A_j(b_k + l) \neq B_k(a_j - l' - 1)$ with $l, l' = 0, 1, 2, \ldots$, $j = 1, \ldots, n$, and $k = 1, \ldots, m$.

Also, the contour $\mathcal{L}$ can be chosen as three possible types $\mathcal{L}_{+\infty}$, $\mathcal{L}_{-\infty}$, $(\gamma - \mathrm{i}\infty, \gamma + \mathrm{i}\infty)$, $\gamma \in \mathbb{R}$. Furthermore, depend on the following parameters

$$\begin{aligned} \rho &= \prod_{j=1}^{p} {A_j}^{-A_j} \prod_{k=1}^{q} {B_k}^{B_k}, \\ \Delta &= \sum_{k=1}^{q} B_k - \sum_{j=1}^{p} A_j, \\ \mu &= \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j + \frac{p-q}{2}, \\ a^* &= \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{k=1}^{m} B_k - \sum_{k=m+1}^{q} B_k, \end{aligned}$$

the choices of the contour $\mathcal{L}$ and convergence domains for analytic function H can be found. For more details about this functions such as convergency, analytic continuation and their applications in applied sciences the reader is referred to references[21–26].

Remark 1 In the presence of a multiple pole s_0 of order n we need to expand the power series of the involved functions at the pole and evaluate the coefficient of the term $1/(s - s_0)$ as the residue. In this case the expansions of z^s and Gamma functions have the forms

$$\begin{aligned} z^s &= z^{s_0}[1 + \log z(s - s_0) + O((s - s_0)^2)], \\ \Gamma(s) &= \Gamma(s_0)[1 + \psi(s_0)(s - s_0) + O((s - s_0)^2)], \\ &\qquad s \to s_0, s_0 \neq 0, -1, -2, \ldots \\ \Gamma(s) &= \frac{(-1)^k}{\Gamma(k+1)(s+k)}[1 + \psi(k+1)(s+k) \\ &\qquad + O((s+k)^2)], \quad s \to -k,\ k = 0, 1, 2, \ldots \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

THE TIME-FRACTIONAL KLEIN-GORDON EQUATION OF DISTRIBUTED ORDER

The following equation is called *the time-fractional Klein-Gordon equation of distributed order* with fractional derivative in the Caputo sense. For the given order-density function $b_1(\alpha)$ and initial and boundary conditions

$$\begin{gathered} \int_1^2 b_1(\alpha)[{}^C_tD^{\alpha}_{0+} u(x,t)]\,\mathrm{d}\alpha - c^2 u_{xx}(x,t) + d^2 u = q(x,t), \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \\ \lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \\ x, c, d \in \mathbb{R}, \quad b_1(\alpha) \geqslant 0, \\ \int_1^2 b_1(\alpha)\,\mathrm{d}\alpha = 1, \quad (3) \end{gathered}$$

the solution of the time-fractional Klein-Gordon equation of distributed order is presented in the following theorem.

Theorem 1 *In view of the above conditions, the following relation is the solution of the time-fractional Klein-Gordon equation of distributed order* (3).

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G_1(x-\xi,t)\,\mathrm{d}\xi - \int_{-\infty}^{\infty} g(\xi) G_2(x-\xi,t)\,\mathrm{d}\xi - \int_0^t q(\xi,\eta) G(x-\xi,t-\eta)\,\mathrm{d}\eta, \quad (4)$$

where the fundamental solutions G_1, G_2 and G_3 (Green's functions) are denoted as

$$G_1(x,t) = -\frac{1}{2\pi x}\int_0^{\infty}\frac{\mathrm{e}^{-rt}}{r}\left[H\left(\frac{\sqrt{\rho}}{x}\right) - \frac{d^2}{c^2}H^*\left(\frac{\sqrt{\rho}}{x}\right)\right]\mathrm{d}r,$$
$$G_2(x,t) = \frac{1}{2\pi x}\int_0^{\infty}\frac{\mathrm{e}^{-rt}}{r^2}\left[H\left(\frac{\sqrt{\rho}}{x}\right) + \frac{d^2}{c^2}H^*\left(\frac{\sqrt{\rho}}{x}\right)\right]\mathrm{d}r,$$
$$G(x,t) = \frac{1}{2\pi c^2}\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{e}^{-rt}\frac{1}{x}H^*\left(\frac{\sqrt{\rho}}{x}\right)\mathrm{d}r\,\mathrm{d}\xi, \quad (5)$$

where ρ, γ are given by relation (13) *and the functions H and H^* are expressed in terms of Fox H function expressed at the end of paper*

$$H\left(\frac{\sqrt{\rho}}{x}\right) = \pi H_{1,2}^{1,0}\left[\frac{\sqrt{\rho}}{x}\left|\begin{matrix} -;(1,\frac{\gamma}{2}) \\ (1,1);(1,\frac{\gamma}{2})\end{matrix}\right.\right], \quad (6)$$
$$H^*\left(\frac{\sqrt{\rho}}{x}\right) = \frac{\pi}{\rho}H_{0,3}^{1,0}\left[\frac{\sqrt{\rho}}{x}\left|\begin{matrix} - \\ (1,1);(\gamma,\frac{\gamma}{2});(1+\gamma,\frac{\gamma}{2})\end{matrix}\right.\right].$$

Proof: In order to obtain the solution of (3), we extend the approach by Naber[27] to find the fundamental solution related to a generic order-density function $b_1(\alpha)$. In this respect, by applying the Laplace transform with respect to t on (3)[28]

$$\mathcal{L}\{{}_t^C D_{0^+}^{\alpha} u(x,t); s\} = s^{\alpha}\tilde{u}(x,s) - s^{\alpha-1}u(x,0^+) - s^{\alpha-2}u_t(x,0^+), \ s \in \mathbb{C}, \quad (7)$$

and the Fourier transform with respect to x[29]

$$\mathcal{F}\{u_{xx}(x,t);k\} = \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}u_{xx}(x,t)\,\mathrm{d}x = -k^2\hat{u}(k,t), \quad k \in \mathbb{R}, \quad (8)$$

we obtain the transformed equation in the form

$$\hat{\tilde{u}}(k,s)\int_1^2 b_1(\alpha)s^{\alpha}\,\mathrm{d}\alpha - F(k)\int_1^2 b_1(\alpha)s^{\alpha-1}\,\mathrm{d}\alpha - G(k)\int_1^2 b_1(\alpha)s^{\alpha-2}\,\mathrm{d}\alpha + c^2k^2\hat{\tilde{u}}(k,s) + d^2\hat{\tilde{u}}(k,s) = \hat{\tilde{q}}(k,s), \quad (9)$$

which can be rewritten again in the following form

$$\hat{\tilde{u}}(k,s) = \frac{F(k)(sB_1(s) - s\frac{d^2}{c^2}) - G(k)(B_1(s) - \frac{d^2}{c^2})}{s^2(B_1(s)+k^2)} + \frac{\hat{\tilde{q}}(k,s)}{c^2(B_1(s)+k^2)}, \quad (10)$$

where $F(k)$ and $G(k)$ is the Fourier transform of the functions $f(x)$ and $g(x)$, respectively, and

$$B_1(s) = \frac{1}{c^2}\left[\int_1^2 b_1(\alpha)s^{\alpha}\,\mathrm{d}\alpha + d^2\right]. \quad (11)$$

Now, by virtue of the Titchmarsh theorem for the inverse Laplace transform[30,31] of functions $\hat{\tilde{u}}_1(k,s) = B_1(s)/s^n(B_1(s)+k^2)$ and $\hat{\tilde{u}}_2(k,s) = 1/s^n(B_1(s)+k^2)$, we have the following relation for $j=1,2$

$$\hat{u}_j(k,t) = -\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-rt}\,\mathrm{Im}\{\hat{u}_j(k,r\,\mathrm{e}^{\mathrm{i}\pi})\}\,\mathrm{d}r. \quad (12)$$

In order to simplify the above relation, we need to evaluate the imaginary part of the functions $-\hat{u}_j(k,r\,\mathrm{e}^{\mathrm{i}\pi})$ along the ray $s = r\,\mathrm{e}^{\mathrm{i}\pi}$, $r>0$, (where is the branch cut of the function s^{α}). In this regard, by writing

$$B_1(r\,\mathrm{e}^{\mathrm{i}\pi}) = \rho\cos\gamma\pi + \mathrm{i}\rho\sin\gamma\pi, \quad \begin{cases} \rho = \rho(r) = |B_1(r\,\mathrm{e}^{\mathrm{i}\pi})| \\ \gamma = \gamma(r) = \frac{1}{\pi}\arg[B_1(r\,\mathrm{e}^{\mathrm{i}\pi})], \end{cases} \quad (13)$$

and evaluating the imaginary part of the functions $-\hat{\tilde{u}}_j$, $j=1,2$ for $n=0,1,2$

$$\mathrm{Im}\left\{\frac{B_1(s)}{s^n(B_1(s)+k^2)}\right\} = K_1(n,k,r) = \frac{k^2\rho\sin(\pi\gamma)}{(-r)^n(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}, \quad (14)$$

$$\mathrm{Im}\left\{\frac{1}{s^n(B_1(s)+k^2)}\right\} = K_2(n,k,r) = \frac{-\rho\sin(\pi\gamma)}{(-r)^n(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}, \quad (15)$$

and substituting in the relation (10), we get

$$\begin{aligned}\hat{u}(k,t) = &-\frac{F(k)}{\pi}\int_0^\infty \mathrm{e}^{-rt}\bigg[K_1(1,k,r) \\ &\qquad -\frac{d^2}{c^2}K_2(1,k,r)\bigg] \\ &-\frac{G(k)}{\pi}\bigg[K_1(2,k,r)-\frac{d^2}{c^2}K_2(2,k,r)\bigg]\,\mathrm{d}r \\ &-\frac{1}{\pi c^2}\int_0^\infty \mathrm{e}^{-rt}[\hat{q}(k,t) *_t K_2(0,k,r)]\,\mathrm{d}r, \end{aligned} \tag{16}$$

where the symbol $*_t$ is the convolution of the Laplace transform. For the Fourier inversion, since the function $\hat{u}(k,t)$ is even in k, the inversion of the above integral takes the form

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \cos(kx)\hat{u}^*(k,t)\,\mathrm{d}k, \tag{17}$$

where the function $\hat{u}^*(k,t)$ is given by

$$\begin{aligned}\hat{u}^*(k,t) = f(x) *_x &\bigg\{-\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\bigg[K_1(1,k,r) \\ &-\frac{d^2}{c^2}K_2(1,k,r)\bigg]\,\mathrm{d}r\bigg\} \\ - g(x) *_x &\bigg\{\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\bigg[K_1(2,k,r) \\ &-\frac{d^2}{c^2}K_2(2,k,r)\bigg]\,\mathrm{d}r\bigg\} \\ - q(x,t) *_t *_x &\bigg\{\frac{1}{\pi c^2}\int_0^\infty \mathrm{e}^{-rt}[K_2(0,k,r)]\,\mathrm{d}r,\bigg\} \end{aligned} \tag{18}$$

and the symbol $*_x$ is the convolution of the Fourier transform. In this stage, for evaluating the Fourier integral (17), we use the Mellin transform for $c_1 < \mathrm{Re}(s) < c_2$

$$\begin{aligned}\mathcal{M}\{f(x);s\} = F(s) &= \int_0^\infty x^{s-1}f(x)\,\mathrm{d}x, \\ f(x) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s)x^{-s}\,\mathrm{d}s, \quad c = \mathrm{Re}(s),\end{aligned} \tag{19}$$

with convolution theorem

$$\mathcal{M}\{a(\xi)*b(\xi)\} = \mathcal{M}\bigg\{\int_0^\infty a(\eta)b\bigg(\frac{\xi}{\eta}\bigg)\frac{\mathrm{d}\eta}{\eta}\bigg\} = A(s)B(s). \tag{20}$$

By identifying the Fourier integral (17) as the Mellin convolution in k and setting $\xi = 1/x, \eta = k$,

$$a(k,t) = \hat{u}^*(k,t), \quad b(k,x) = \frac{1}{\pi kx}\cos\bigg(\frac{1}{k}\bigg),$$

the explicit solution $u(x,t)$ can be written as the Mellin inversion of product $A(s,t)B(s,x)$, namely

$$u(x,t) = \frac{1}{x}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A(s,t)B(s,x)x^{-s}\,\mathrm{d}s, \tag{21}$$

where $B(s,x)$ can be obtained from the handbook by Erdelyi et al[32] as follows

$$B(s,x) = \frac{\Gamma(1-s)}{x\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}, \quad 0 < \mathrm{Re}(s) < 1. \tag{22}$$

For the required Mellin Transform of $A(s,t)$

$$A(s,t) = \int_0^\infty \hat{u}^*(k,t)k^{s-1}\,\mathrm{d}k, \tag{23}$$

which depend on the terms of brackets in (18)

$$\begin{aligned} &A(s,t) \\ &= \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\bigg\{\frac{1}{\pi}\int_0^\infty \bigg[\frac{k^2\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)} \\ &\quad + \frac{d^2}{c^2}\frac{\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}\bigg]k^{s-1}\,\mathrm{d}k\bigg\}\,\mathrm{d}r \\ &\quad + \int_0^\infty \frac{\mathrm{e}^{-rt}}{r^2}\bigg\{\frac{1}{\pi}\int_0^\infty \bigg[\frac{k^2\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)} \\ &\quad - \frac{d^2}{c^2}\frac{\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}\bigg]k^{s-1}\,\mathrm{d}k\bigg\}\,\mathrm{d}r \\ &\quad + \frac{1}{\pi c^2}\int_0^\infty \mathrm{e}^{-rt}\bigg\{\frac{1}{\pi} \\ &\qquad \int_0^\infty \frac{\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}k^{s-1}\,\mathrm{d}k\bigg\}\mathrm{d}r, \end{aligned} \tag{24}$$

we use the change of variable $k^2 = \rho y$ and apply the following integral[32]

$$\begin{aligned} &\frac{1}{\pi}\int_0^\infty \frac{\sin(\pi\gamma)}{y^2+2y\rho\cos(\pi\gamma)+1}y^{s-1}\,\mathrm{d}y \\ &\qquad = -\frac{\sin((s-1)\pi\gamma)}{\sin(\pi s)} \\ &\qquad = -\frac{\pi\Gamma(s)\Gamma(1-s)}{\Gamma(\gamma(s-1))\Gamma(1-\gamma(s-1))} \\ &|\gamma| < 1, 0 < \mathrm{Re}(s) < 2, \end{aligned} \tag{25}$$

to simplify the relation (24) into

$$
\begin{aligned}
&A(s,t)\\
&=\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\bigg\{-\frac{\rho^{s/2}}{2}\frac{\Gamma(1+\frac{s}{2})\Gamma(1-(1+\frac{s}{2}))}{\Gamma(\frac{\gamma s}{2})\Gamma(1-\frac{\gamma s}{2})}\\
&\quad -\frac{d^2}{c^2}\frac{\rho^{(s/2)-1}}{2}\frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(\gamma(\frac{s}{2}-1))\Gamma(1-\gamma(1-\frac{s}{2}))}\bigg\}\,\mathrm{d}r\\
&\quad +\int_0^\infty \frac{\mathrm{e}^{-rt}}{r^2}\bigg\{-\frac{\rho^{s/2}}{2}\frac{\Gamma(1+\frac{s}{2})\Gamma(1-(1+\frac{s}{2}))}{\Gamma(\frac{\gamma s}{2})\Gamma(1-\frac{\gamma s}{2})}\\
&\quad +\frac{d^2}{c^2}\frac{\rho^{(s/2)-1}}{2}\frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(\gamma(\frac{s}{2}-1))\Gamma(1-\gamma(1-\frac{s}{2}))}\bigg\}\,\mathrm{d}r\\
&\quad -\int_0^\infty \mathrm{e}^{-rt}\bigg\{\frac{\rho^{(s/2)-1}}{2}\frac{\Gamma(\frac{s}{2})}{\Gamma(\gamma(\frac{s}{2}-1))}\\
&\quad \frac{\Gamma(1-\frac{s}{2})}{\Gamma(1-\gamma(1-\frac{s}{2}))}\bigg\}\,\mathrm{d}r.
\end{aligned}
\tag{26}
$$

Finally, by substituting (24) into (21), we can write the solution $u(x,t)$ with respect to Green's functions in the forms of relations (4), (5) and associated Fox H functions. □

Time fractional wave equation of single order

When we set $b(\beta)=\delta(\beta-n)$, $1<n<2$, $d=0$, (where δ is the Dirac delta function) the relation (3) is converted to time fractional wave equation of single order n

$$ {}_t^C D_{0+}^n u(x,t)-c^2u_{xx}(x,t)=q(x,t), \tag{27}$$

and (11) and (13) are written as

$$B(s)=\frac{1}{c^2}s^n,\ \rho=\rho(r)=\frac{1}{c^2}r^n,\ \gamma=n.$$

Also, the transformed equation $\hat{\hat{u}}(k,s)$ in (10), takes the form

$$\hat{\hat{u}}(k,s)=\frac{s^{n-1}F(k)+s^{n-2}G(k)+\hat{\hat{q}}(k,s)}{s^n+\frac{1}{c^2}k^2}. \tag{28}$$

Since, the inverse Laplace transform of $(s^{n-m})/(s^n+(1/c^2)k^2)$ in (28) can be easily obtained as the Mittag-Leffler functions of order n[28]

$$
\begin{aligned}
\mathcal{L}^{-1}\bigg\{\frac{s^{n-m}}{s^n+k^2/c^2}\bigg\}&=t^{m-1}E_n\bigg(-\frac{k^2t^n}{c^2}\bigg)\\
&=t^{m-1}\sum_{j=0}^{\infty}\frac{(-k^2t^n/c^2)^j}{\Gamma(nj+1)},
\end{aligned}
$$

the remaining solution with respect to Fourier inversion can be written as follows

$$
\begin{aligned}
&u(x,t)\\
&=f(x)*_x\frac{1}{2\pi}\int_{-\infty}^{\infty}\bigg[E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg]\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k\\
&\quad+g(x)*_x\frac{1}{2\pi}\int_{-\infty}^{\infty}\bigg[tE_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg]\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k\\
&\quad+q(x,t)*_x*_t\frac{1}{2\pi}\int_{-\infty}^{\infty}\bigg[t^nE_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg]\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k,
\end{aligned}
\tag{29}
$$

where $*_x, *_t$ is the convolutions of the Fourier and Laplace transforms, respectively. To calculate the above integrals by writing Fourier kernel in real and imaginary part according to the following relation

$$
\begin{aligned}
\mathcal{F}^{-1}[K(k);x]&=\frac{1}{\pi}\int_0^\infty\cos(kx)\,\mathrm{Re}(K(k))\,\mathrm{d}k\\
&=\frac{1}{\pi}\int_0^\infty\sin(kx)\,\mathrm{Im}(K(k))\,\mathrm{d}k,
\end{aligned}
\tag{30}
$$

the explicit solution gives rise to

$$
\begin{aligned}
&u(x,t)\\
&=f(x)*_x\bigg[\frac{1}{\pi}\int_0^\infty\cos(kx)\,\mathrm{Re}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\\
&\quad+\frac{1}{\pi}\sin(kx)\,\mathrm{Im}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\,\mathrm{d}k\bigg]\\
&\quad+g(x)*_x\bigg[\frac{t}{\pi}\int_0^\infty\cos(kx)\,\mathrm{Re}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\\
&\quad+\frac{t}{\pi}\sin(kx)\,\mathrm{Im}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\,\mathrm{d}k\bigg]\quad.\\
&\quad+q(x,t)*_x*_t\bigg[\frac{t^n}{\pi}\int_0^\infty\cos(kx)\\
&\quad\times\mathrm{Re}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\\
&\quad+\frac{t^n}{\pi}\sin(kx)\,\mathrm{Im}\bigg(E_n\bigg(-\frac{1}{c^2}k^2t^n\bigg)\bigg)\,\mathrm{d}k\bigg].
\end{aligned}
\tag{31}
$$

To change the above relation in the Mellin convolution (20), in a same procedure to pervious section we use the following facts[32]

$$\begin{aligned} b(k,x) &= \frac{1}{\pi k x}\cos\left(\frac{1}{k}\right),\\ &\overset{\mathcal{M}}{\to} B(s,x) = \frac{\Gamma(1-s)}{\pi x}\sin\left(\frac{\pi s}{2}\right),\\ &0<\mathrm{Re}(s)<1,\\ b(k,x) &= \frac{1}{\pi k x}\sin\left(\frac{1}{k}\right),\\ &\overset{\mathcal{M}}{\to} B(s,x) = \frac{\Gamma(1-s)}{\pi x}\cos\left(\frac{\pi s}{2}\right),\\ &0<\mathrm{Re}(s)<2,\\ a(k,t) &= E_n\left(-\frac{t^n}{c^2}k^2\right),\\ &\overset{\mathcal{M}}{\to} A(s,x) = \frac{1}{2}\frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(1-\frac{ns}{2})}, \end{aligned}$$

to get the explicit solution $u(x,t)$ with respect to Green's function in terms of the Mellin-Barnes integral as follows

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} f(\xi)G(x-\xi,t)\,\mathrm{d}\xi\\ &\quad - t\int_{-\infty}^{\infty} g(\xi)G(x-\xi,t)\,\mathrm{d}\xi\\ &\quad - \int_0^t \eta^n q(\xi,\eta)G(x-\xi,t-\eta)\,\mathrm{d}\eta, \end{aligned} \tag{32}$$

where Green's function is given by

$$\begin{aligned} G(x,t) &= \frac{1}{4\pi^2 \mathrm{i}x}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\Gamma(1-s)}{\Gamma(1-\frac{ns}{2})}\\ &\qquad \sin\left(\frac{s\pi}{2}\right)\left(\frac{1}{x}\right)^s \mathrm{d}s\\ &= \frac{1}{2x}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\Gamma(1-s)}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})\Gamma(1-\frac{ns}{2})}\\ &\qquad \left(\frac{1}{x}\right)^s \mathrm{d}s, \end{aligned} \tag{33}$$

or equivalently

$$G(x,t) = \frac{1}{2x}H_{3,3}^{1,2}\left[\frac{1}{x}\,\middle|\, \begin{matrix}(1,\frac{1}{2});(1,\frac{n}{2});(1,\frac{1}{2})\\(1,\frac{1}{2});(1,1);(1,\frac{1}{2})\end{matrix}\right].$$

Green's function (34) is similar to the fundamental solution of the space-time fractional diffusion equation in paper of Mainardi et al[8].

Another form of the explicit solution

Also, another form of the explicit solution (4) can be written in a closed form using the direct Fourier-Cosine transform with respect to k in (17). For this purpose, we use the following identities[32]

$$\begin{aligned} K_1^*(x,r) &= \mathcal{F}_C\left\{\frac{k^2\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)};x\right\}\\ &= \frac{\pi}{2\sqrt{\rho}}\frac{1}{\sin^2(\pi\gamma)}\,\mathrm{e}^{-\sqrt{\rho}x\cos\left((\pi\gamma/2)\right)}\\ &\qquad \sin\left(\frac{\pi\gamma}{2}-\sqrt{\rho}x\sin\left(\frac{\pi\gamma}{2}\right)\right),\\ K_2^*(x,r) &= \mathcal{F}_C\left\{\frac{-\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)};x\right\}\\ &= -\frac{\pi}{2\sqrt{\rho^3}}\frac{1}{\sin^2(\pi\gamma)}\,\mathrm{e}^{-\sqrt{\rho}x\cos\left((\pi\gamma/2)\right)}\\ &\qquad \sin\left(\frac{\pi\gamma}{2}+\sqrt{\rho}x\sin\left(\frac{\pi\gamma}{2}\right)\right), \end{aligned} \tag{34}$$

to write the relation (17) into

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} f(\xi)G_1(x-\xi,t)\,\mathrm{d}\xi\\ &\quad - \int_{-\infty}^{\infty} g(\xi)G_2(x-\xi,t)\,\mathrm{d}\xi\\ &\quad - \int_0^t q(\xi,\eta)G(x-\xi,t-\eta)\,\mathrm{d}\eta, \end{aligned} \tag{35}$$

where Green's functions G_1, G_2 and G_3 are denoted as

$$\begin{aligned} G_1(x,t) &= \frac{-1}{2\pi}\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\left[K_1^*(x,r)-\frac{d^2}{c^2}K_2^*(x,r)\right]\mathrm{d}r,\\ G_2(x,t) &= \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{e}^{-rt}}{r^2}\left[K_1^*(x,r)+\frac{d^2}{c^2}K_2^*(x,r)\right]\mathrm{d}r,\\ G(x,t) &= \frac{1}{2\pi c^2}\int_{-\infty}^{\infty}\int_0^\infty \mathrm{e}^{-rt}K_2^*(x,r)\,\mathrm{d}r\,\mathrm{d}\xi, \end{aligned} \tag{36}$$

provided that the integrals on the right-hand side of (35) are convergent.

THE TIME-FRACTIONAL FOKKER-PLANCK EQUATION OF DISTRIBUTED ORDER

In this section, we study *the time-fractional Fokker-Planck equation of distributed order* with fractional derivative in the Caputo sense. This equation is a generalization of Hassan's equation[33] with the order-density function $b_2(\alpha)$ and the initial and boundary conditions

$$\begin{aligned}
&\int_0^1 b_2(\alpha)[{}_t^C D_{0+}^{\alpha} u(r,t)]\,\mathrm{d}\alpha \\
&\qquad = u_{rr}(r,t) + \frac{1}{r}u_r - \lambda u + s(r) \\
&u(r,0) = f(r), \qquad \lim_{|r|\to\infty} u(r,t) < \infty, \\
&t > 0, r \geqslant 0, b_2(\alpha) \geqslant 0, \qquad \int_0^1 b_2(\alpha)\,\mathrm{d}\alpha = 1.
\end{aligned} \tag{37}$$

The solution of the time-fractional Fokker-Planck equation of distributed order is presented in the following theorem.

Theorem 2 *In view of the above conditions, there holds the following formula for the solution of the time-fractional Fokker-Planck equation of distributed order* (37)

$$u(r,t) = \int_0^\infty \left[-\frac{f(r)}{r}G_1(r,t) + \frac{s(r)}{r}G_2(r,t)\right]\mathrm{d}r, \tag{38}$$

where Green's functions G_1 and G_2 are given by

$$\begin{aligned}
G_1 &:= \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} \\
&\quad H_{3,6}^{2,2}\left[\frac{\sqrt{\rho}}{2r}\,\middle|\, \begin{matrix}(0,\frac{1}{2});(2,1);(1,\frac{\gamma}{2})\\ (0,\frac{1}{2})_2;(1,1)_3;(1,\frac{\gamma}{2})\end{matrix}\right]\mathrm{d}r', \\
G_2 &:= \frac{1}{\rho}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} \\
&\quad H_{2,7}^{2,2}\left[\frac{\sqrt{\rho}}{2r}\,\middle|\, \begin{matrix}(1,\frac{1}{2});(2,1)\\ (1,\frac{1}{2})_2;(1,1)_3;(1+\gamma,\frac{\gamma}{2});(\gamma,\frac{\gamma}{2})\end{matrix}\right]\mathrm{d}r',
\end{aligned}$$

and ρ, γ are shown by the relation (43) *and $(a,b)_n$ is denoted as n iterations of (a,b).*

Proof: In order to obtain the solution of (37) using the Laplace transform with respect to t

$$\mathcal{L}\{{}_t^C D_{0+}^{\alpha} u(r,t); s\} = s^\alpha \tilde{u}(x,s) - s^{\alpha-1}u(x,0^+), \tag{39}$$

and the Hankel transform of zero order with respect to r[29]

$$\begin{aligned}
\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r; k\right\} &= \int_0^\infty rJ_0(rk)\left[u_{rr} + \frac{1}{r}u_r\right]\mathrm{d}r, \\
&= -k^2\hat{u}(k,t),
\end{aligned} \tag{40}$$

we obtain for $s \in \mathbb{C}$, $k > 0$

$$\hat{\tilde{u}}(k,s) = \frac{B_2(s)F(k) + S(k)}{s(B_2(s) + k^2)}, \tag{41}$$

where $F(k)$ and $S(k)$ is the Hankel transform of the functions $f(r)$ and $s(r)$, respectively, and

$$B_2(s) = \int_0^1 b_2(\alpha)s^\alpha\,\mathrm{d}\alpha + \lambda, \tag{42}$$

$$\begin{aligned}
&B_2(r'\,\mathrm{e}^{\mathrm{i}\pi}) = \rho\cos\gamma\pi + \mathrm{i}\rho\sin\gamma\pi, \\
&\qquad\left\{\begin{array}{l}\rho = \rho(r') = \left|B_1(r'\,\mathrm{e}^{\mathrm{i}\pi})\right| \\ \gamma = \gamma(r') = \frac{1}{\pi}\arg[B_1(r'\,\mathrm{e}^{\mathrm{i}\pi})].\end{array}\right.
\end{aligned} \tag{43}$$

By the same procedure to the previous problem for the Laplace inversion via the Titchmarsh theorem, we have

$$\hat{u}(k,t) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\frac{\rho\sin(\pi\gamma)[k^2F(k) - S(k)]}{k^4 + 2k^2\rho\cos(\pi\gamma) + \rho^2}\,\mathrm{d}r'. \tag{44}$$

For the inverse Hankel transform, we use its definition

$$u(r,t) = \int_0^\infty kJ_0(kr)\hat{u}(k,t)\,\mathrm{d}k, \tag{45}$$

and apply Parseval's identity for this transform[29]

$$\begin{aligned}
\int_0^\infty rf(r)g(r)\,\mathrm{d}r = \int_0^\infty & k\mathcal{H}_0\{f(r);k\} \\
&\mathcal{H}_0\{g(r);k\}\,\mathrm{d}k,
\end{aligned} \tag{46}$$

to reduce (44) to

$$\begin{aligned}
u(r,t) &= \frac{1}{\pi}\int_0^\infty rf(r) \\
&\quad \mathcal{H}_0^{-1}\left[\frac{J_0(kr)k^2\rho\sin(\pi\gamma)}{(k^4 + 2k^2\rho\cos(\pi\gamma) + \rho^2)}\right]\mathrm{d}r \\
&\quad \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r' \\
&\quad - \frac{1}{\pi}\int_0^\infty rs(r) \\
&\quad \mathcal{H}_0^{-1}\left[\frac{J_0(kr)\rho\sin(\pi\gamma)}{(k^4 + 2k^2\rho\cos(\pi\gamma) + \rho^2)}\right]\mathrm{d}r \\
&\quad \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r'.
\end{aligned} \tag{47}$$

As a consequence of the above equation by considering the inverse Hankel transform as the Mellin transform convolution (20) and setting

$$\begin{aligned}
a(k) &= \left[\frac{\binom{k^2}{1}\rho\sin(\pi\gamma)}{(k^4 + 2k^2\rho\cos(\pi\gamma) + \rho^2)}\right], \\
&= b(k,r) = \frac{1}{k^2r^2}J_0^2\left(\frac{1}{k}\right), \quad \xi = \frac{1}{r}, \eta = k,
\end{aligned}$$

the (47) is converted to

$$u(r,t)=\frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r'\int_0^\infty \frac{f(r)}{r}\,\mathrm{d}r \left\{\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathcal{M}\left\{\frac{k^2\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}\right\} \mathcal{M}\left\{\frac{1}{k^2}J_0^2\left(\frac{1}{k}\right)\right\}r^{-s}\,\mathrm{d}s\right\} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r'\int_0^\infty \frac{s(r)}{r}\,\mathrm{d}r \left\{\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathcal{M}\left\{\frac{\rho\sin(\pi\gamma)}{(k^4+2k^2\rho\cos(\pi\gamma)+\rho^2)}\right\} \mathcal{M}\left\{\frac{1}{k^2}J_0^2\left(\frac{1}{k}\right)\right\}r^{-s}\,\mathrm{d}s\right\}. \quad (48)$$

Now, by substituting the Mellin transform of $(1/k^2)J_0^2(1/k)$[32]

$$\mathcal{M}\left\{\frac{1}{k^2}J_0^2\left(\frac{1}{k}\right)\right\}=\frac{2^{-s+1}\Gamma(s-1)\Gamma(-\frac{s}{2}+1)}{\Gamma^3(s)}, \quad 0<\mathrm{Re}(s)<1, \quad (49)$$

the desired explicit solution $u(r,t)$ is given by

$$u(r,t)= -\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r'\int_0^\infty \frac{f(r)}{r}\,\mathrm{d}r \left\{\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\Gamma(1+\frac{s}{2})\Gamma(-\frac{s}{2})}{\Gamma^3(s)} \frac{\Gamma(s-1)\Gamma(-\frac{s}{2}+1)}{\Gamma(\frac{\gamma s}{2})\Gamma(1-\frac{\gamma s}{2})}\left(\frac{\sqrt{\rho}}{2r}\right)^s\mathrm{d}s\right\} + \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{d}r'\int_0^\infty \frac{s(r)}{r}\,\mathrm{d}r \left\{\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{1}{\rho}\frac{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(\gamma(\frac{s}{2}-1))} \frac{\Gamma(s-1)\Gamma(-\frac{s}{2}+1)}{\Gamma(1-\gamma(1-\frac{s}{2}))\Gamma^3(s)}\left(\frac{\sqrt{\rho}}{2r}\right)^s\mathrm{d}s\right\}, \quad (50)$$

which can be rewritten with respect to Fox H functions as follows

$$u(r,t)=\int_0^\infty -\frac{f(r)}{r}G_1(r,t)+\frac{s(r)}{r}G_2(r,t)\,\mathrm{d}r, \quad (51)$$

with Green's functions

$$G_1 := \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} H_{3,6}^{2,2}\left[\frac{\sqrt{\rho}}{2r}\,\middle|\,\begin{matrix}(0,\frac{1}{2});(2,1);(1,\frac{\gamma}{2})\\(0,\frac{1}{2});(1,\frac{1}{2});(1,1)_3;(1,\frac{\gamma}{2})\end{matrix}\right]\mathrm{d}r',$$

$$G_2 := \frac{1}{\rho}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} H_{2,7}^{2,2}\left[\frac{\sqrt{\rho}}{2r}\,\middle|\,\begin{matrix}(1,\frac{1}{2});(2,1)\\(1,\frac{1}{2})_2;(1,1)_3;(1+\gamma,\frac{\gamma}{2});(\gamma,\frac{\gamma}{2})\end{matrix}\right]\mathrm{d}r',$$

provided that the integrals on the right-hand side of (51) are convergent. □

THE TIME-FRACTIONAL GIONA-ROMAN DIFFUSION EQUATION OF DISTRIBUTED ORDER

In connection with initial-value problems in fractals, Giona and Roman[34,35] stated a partial fractional differential equation with non-constant coefficients. The distributed-order generalization of this equation in time can be considered as

$$\int_0^{1/2} b_3(\alpha)[{}_t^C D_{0+}^\alpha u(x,t)]\,\mathrm{d}\alpha = -Cx^{-\beta}u_x(x,t), \quad (52)$$

$$u(x,0)=f(x), u(0,t)=0,$$

$$C>0, \beta\geqslant 0, t>0, x\geqslant 0, b_3(\alpha)\geqslant 0, \int_0^{1/2} b_3(\alpha)\,\mathrm{d}\alpha = 1,$$

with Cauchy type initial and boundary conditions and order-density function $b_3(\alpha)$.

Theorem 3 *In view of the above conditions, the following relation is the solution of the time-fractional Giona-Roman diffusion equation of distributed order* (52).

$$u(x,t)=\int_0^x u^\beta G^\alpha(x^{\beta+1}-u^{\beta+1},t)f(u)\,\mathrm{d}u, \quad (53)$$

where Green's function G^α is given in terms of the Fox H functions as

$$G^\alpha(x,t)=\sqrt{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} H_{0,1}^{1,0}\left[\rho\cos(\pi\gamma)x\,\middle|\,\begin{matrix}-\\(0,1)\end{matrix}\right] \times\left[\tfrac{1}{2}\rho^2\sin^2(\pi\gamma) H_{0,2}^{1,0}\left[\tfrac{1}{4}\rho^2\sin^2(\pi\gamma)x^2\,\middle|\,\begin{matrix}-\\(\frac{1}{2},1);(0,1)\end{matrix}\right] - \rho\sin(\pi\gamma) H_{0,2}^{1,0}\left[\tfrac{1}{4}\rho^2\sin^2(\pi\gamma)x^2\,\middle|\,\begin{matrix}-\\(0,1);(\frac{1}{2},1)\end{matrix}\right]\right]\mathrm{d}r'.$$

Proof: In order to obtain the solution of (52), using the Laplace transform in time t

$$\mathcal{L}\{{}_t^C D_{0+}^\alpha u(r,t);s\}=s^\alpha\tilde{u}(x,s)-s^{\alpha-1}u(x,0^+), \quad (54)$$

and the $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform in space x [19]

$$\mathcal{L}_{((x^{\beta+1})/(\beta+1))}\{u(x,t);p\} = \hat{u}(p,t) = \int_0^\infty x^\beta \, \mathrm{e}^{-p^{\beta+1}\frac{x^{\beta+1}}{\beta+1}} u(x,t)\,\mathrm{d}x, \quad \mathrm{Re}\, p^{\beta+1} > 0, \tag{55}$$

and using the $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform of delta-derivative $\delta_x = x^{-\beta}\mathrm{d}/\mathrm{d}x$ [19],

$$\mathcal{L}_{(x^{\beta+1})/(\beta+1)}\{x^{-\beta}u(x,t);p\} = p^{\beta+1}\hat{u}(p,t) - u(0,t), \tag{56}$$

we obtain the transformed equation in the following form

$$\hat{\hat{u}}(p,s) = \frac{B_3(s)}{C^2 s(B_3(s)+p^{\beta+1})}F(p), \tag{57}$$

where $F(p)$ is the $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform of the function $f(x)$ and the function $B_3(s)$ is given by

$$B_3(s) = C\int_0^{\frac{1}{2}} b_3(\alpha)s^\alpha \,\mathrm{d}\alpha. \tag{58}$$

As with the previous problems for the Laplace inversion via the Titchmarsh theorem, we have

$$\hat{u}(p,t) = -\frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} \left[\frac{p^{\beta+1}\rho\sin(\pi\gamma)}{p^{2\beta+2}+2p^{\beta+1}\rho\cos(\pi\gamma)+\rho^2}F(p)\right]\mathrm{d}r'.$$

For the inversion of the $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform, using the complex inversion formula [20]

$$u(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \hat{u}(\sqrt[\beta+1]{p},t)\,\mathrm{e}^{p((x^{\beta+1})/(\beta+1))}\,\mathrm{d}p, \tag{59}$$

we get the solution $u(x,t)$ in the form

$$u(x,t) = -\frac{1}{\pi}f(x) *_x \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} \left[\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{p\rho\sin(\pi\gamma)}{p^2+2p\rho\cos(\pi\gamma)+\rho^2}, \mathrm{e}^{p((x^{\beta+1})/(\beta+1))}\,\mathrm{d}p\right]\mathrm{d}r, \tag{60}$$

where in the above relation the symbol $*_x$ is convolution of the $\mathcal{L}_{((x^{\beta+1})/(\beta+1))}$ transform and is defined by the following integral [20]

$$f{*}g = \int_0^x u^\beta g(u) f(\sqrt[\beta+1]{x^{\beta+1}-u^{\beta+1}})\,\mathrm{d}u. \tag{61}$$

After evaluating the Bromwich integral in (60), we finally get

$$u(x,t) = \int_0^x u^\beta G^\alpha(x^{\beta+1}-u^{\beta+1},t)f(u)\,\mathrm{d}u, \tag{62}$$

where Green's function G^α is given by

$$G^\alpha(x,t) = \int_0^\infty \frac{\mathrm{e}^{-r't}}{r'}\,\mathrm{e}^{-\rho\cos(\pi\gamma)x} \times [\tfrac{1}{2}\rho^2\sin^2(\pi\gamma)\sin(\rho\sin(\pi\gamma)x) - \rho\sin(\pi\gamma)\cos(\rho\sin(\pi\gamma)x)]\,\mathrm{d}r', \tag{63}$$

or in terms of the Fox H functions

$$G^\alpha(x,t) = \sqrt{\pi}\int_0^\infty \frac{\mathrm{e}^{-r't}}{r'} H_{0,1}^{1,0}\left[\rho\cos(\pi\gamma)x \,\middle|\, \begin{matrix} - \\ (0,1)\end{matrix}\right] \times \left[\tfrac{1}{2}\rho^2\sin^2(\pi\gamma) H_{0,2}^{1,0}\left[\tfrac{1}{4}\rho^2\sin^2(\pi\gamma)x^2 \,\middle|\, \begin{matrix} - \\ (\tfrac{1}{2},1);(0,1)\end{matrix}\right] - \rho\sin(\pi\gamma) H_{0,2}^{1,0}\left[\tfrac{1}{4}\rho^2\sin^2(\pi\gamma)x^2 \,\middle|\, \begin{matrix} - \\ (0,1);(\tfrac{1}{2},1)\end{matrix}\right]\right]\mathrm{d}r',$$

provided that the integral on the right-hand side of the above relation is convergent. □

CONCLUSIONS

In this paper, we have paid special attention to transform methods for finding the fundamental solutions (or the corresponding Green's function) of linear partial time-fractional diffusion equations of distributed order. We have stressed the importance of the Fourier, Laplace, Hankel, and Mellin transforms as analytical approaches for finding exact solutions to these types of equations. The Mellin transform is a supplementary tool to write the transformed equations as Mellin-Barnes integrals and writing the Fox H functions as the proper and well-suited functions in the solutions of these equations.

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