On the non-abelian tensor square of groups of order $p^4$ where $p$ is an odd prime

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ABSTRACT: In this paper, we determine the non-abelian tensor square, $G \otimes G$, for non-abelian groups of order $p^4$, where $p$ is an odd prime.

KEYWORDS: $p$-groups

INTRODUCTION

The non-abelian tensor products have their roots in algebraic $K$-theory as well as in homotopy theory and were introduced by Brown and Loday. The non-abelian tensor square of a group $G$, denoted as $G' \otimes G$, is generated by $gg' \otimes h = (g' \otimes g)h = g \otimes hh'$, where $h, g' \in G$, and $h' = ghg^{-1}$ denotes the conjugate of $g$ by $h$. In 1911, Burnside obtained the classification of groups of order $p^5$. Jang Oh proved that non-abelian groups of order $p^4$ satisfy the conditions in the following theorem.

Theorem 1 Let $G$ be a non-abelian group of order $p^4$. Then one of the following holds.
(i) $|Z(G)| = p^2$, $|G'| = p$, and $G' \subseteq Z(G)$
(ii) $|Z(G)| = p$, $|G'| = p^2$, and $Z(G) \subseteq G'$

In this paper, we focus on the non-abelian groups of order $p^4$ that satisfy the conditions in Theorem 1(i).

Theorem 2 Let $G$ be a group of order $p^4$, where $p$ is an odd prime. Then $G$ is isomorphic to exactly one group in the following list.

1. $G_1 = \langle x, y \mid x^{p^3} = y^p = 1, x^y = x^{1+p^2} \rangle$
2. $G_2 = \langle x, y, z \mid x^p = y^p = z^{p^2} = 1, [x, y] = [y, z] = 1, [x, y] = z^p \rangle$
3. $G_3 = \langle x, y \mid x^{p^2} = y^p = z = 1, x^y = x^{1+p^2} \rangle$
4. $G_4 = M_p \times \langle w \rangle$
5. $G_5 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, y] = [y, z] = 1, [x, y] = z^p \rangle$

where $M_p = \langle x, y \mid x^{p^2} = y^p = z^p = 1, [x, y] = [y, z] = 1, [x, y] = z \rangle$

$\lambda = \langle w \mid w^p = 1 \rangle, [x, y] = [y, z] = 1, and [x, y] = z^p$.

PRELIMINARIES

This section includes some basic results on the Schur multiplier and non-abelian tensor square of groups which are used in order to prove our main theorem.

In 2001, Seon Ok obtained the Schur multiplier of groups of order $p^4$, where $p$ is an odd prime as stated in the following theorem.

Theorem 3 Let $G$ be groups of order $p^4$, where $p$ is an odd prime. Then exactly one of the following holds:

$M(G) = \begin{cases} 1, & G \text{ is } G_1 \\ (\mathbb{Z}_p)^2, & G \text{ is } G_2, G_5 \text{ or } G_6 \\ \mathbb{Z}_p, & G \text{ is } G_3 \\ (\mathbb{Z}_p)^4, & G \text{ is } G_4 \end{cases}$

The following five theorems stated are used to compute the non-abelian tensor square of some finite groups.
In 1987, Brown et al. computed the non-abelian tensor square of some groups such as quaternion groups, dihedral groups, symmetric groups and metacyclic groups. The non-abelian tensor square of metacyclic group is presented in the following theorem.

Theorem 4 If \( G = \langle x, y | y^n = x^m = 1, xyx^{-1} = y^l \rangle \) where \( l^m \equiv 1 \pmod{n} \) and \( n \) is an odd number, then \( G \otimes G = \mathbb{Z}_m \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \) where \( m_1 = (n, l - 1) \), \( m_2 = (n, l - 1, l + \cdots + l^{m-1}) \), \( m_3 = (n, 1 + l + \cdots + l^{m-1}) \).

Brown et al. also computed the non-abelian tensor square of direct product of two groups. In this case, the non-abelian tensor square can be computed by the use of Theorem 5. They also determined two properties for Whitehead’s universal quadratic functor \( \Gamma \) as stated in Theorem 6.

Theorem 5 Let \( G \) and \( H \) be groups. Then \((G \times H) \otimes (G \times H) \cong (G \otimes G) \times (G \otimes H) \times (H \otimes G) \times (H \otimes H)\).

Theorem 6 Let \( G \) and \( H \) be abelian groups. Then (i) \( \Gamma(G \times H) = \Gamma(G) \times \Gamma[H] \Gamma(G \otimes H) \), (ii) \( \Gamma \mathbb{Z}_n = \begin{cases} \mathbb{Z}_n, & n \text{ is odd} \\ \mathbb{Z}_{2n}, & n \text{ is even} \end{cases} \).

In the following theorem, Blyth et al. computed the non-abelian tensor square of group \( G \) with \( G^{ab} \) which is finitely generated.

Theorem 7 Let \( G \) be a group such that \( G^{ab} \) is finitely generated. If \( G^{ab} \) has no element of order two or if \( G' \) has no complement in \( G \) then \( G \otimes G \cong \Gamma(G^{ab}) \times G \wedge G \).

Nakaoka gives the conditions that can be used to compute the non-abelian tensor of a finite group.

Theorem 8 Let \( G \) be a finite group and \( i \geq 0 \). Then (i) there is an exact sequence

\[ 1 \rightarrow [G_{i+1}, G^{ab}_i] \rightarrow \tau(G_i, G_i) \rightarrow \tau(G_i^{ab}, G_i^{ab}) \rightarrow 1 \]

where \([G_{i+1}, G^{ab}_i] \leq \tau(G_i, G_i)\).

(ii) \( |G_i \otimes G_i| \leq |G^{ab}_i \otimes G^{ab}_i| |G_{i+1} \otimes G_i| \).

The Schur multiplier, non-abelian tensor square and capability of groups of order \( p^2 \) have been considered by Rashid et al. in Ref. 9, where \( p \) and \( q \) are distinct primes. In Ref. 10, they also computed the Schur multiplier of groups of order \( 8q \), where \( q \) is an odd prime.

Proof of Main Theorem

In this paper, we focus on the non-abelian tensor square of non-abelian group of order \( p^3 \), where \( p \) is an odd prime. The non-abelian tensor square of groups of order \( p^4 \), where \( p \) is an odd prime is computed in the next theorem.

Theorem 9 Let \( G \) be a group of order \( p^4 \), where \( p \) is an odd prime. Then

\[
G \otimes G = \begin{cases}
\mathbb{Z}_{p^2} \times (\mathbb{Z}_p)^3, & G \text{ is } G_1, \\
(\mathbb{Z}_p)^9, & G \text{ is } G_2 \text{ or } G_5,
\end{cases}

(\mathbb{Z}_p)^{11}, & G \text{ is } G_4,
\mathbb{Z}_{p^2} \times (\mathbb{Z}_p)^5, & G \text{ is } G_6.
\]

Proof: Let \( G \) be a non-abelian group of order \( p^4 \), where \( p \) is an odd prime. By Theorem 2 there are 6 types of these groups. First, we prove for \( G_1 \), by choosing \( n = p^3 \), \( m = p \), \( l = 1 + p^2 \), and \( G_1 \) is a metacyclic group. Then by Theorem 4, \( G \otimes G = \mathbb{Z}_m \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \) where \( m_1 = (p^3, 1 + p^2 - 1) = p^2 \), \( m_2 = (p^3, 1 + p^2 - 1, 1 + (1 + p^2) + \cdots + (1 + p^2)^{p-1}) = p \), and \( m_3 = (p^3, 1 + (1 + p^2) + \cdots + (1 + p^2)^{p-1}) = p \). Therefore \( G \otimes G \cong \mathbb{Z}_{p^2} \times (\mathbb{Z}_p)^3 \).

For group \( G_2 \), by Theorem 8, the following computations are considered. In this group, \( G_2 = \mathbb{Z}_p \) and \( G_2^{ab} = (\mathbb{Z}_p)^3 \). As \( G_2^{ab} \otimes G_2^{ab} \cong (\mathbb{Z}_p)^9 \), \( G_2^{ab} \cong 1 \) and \( G_2 \otimes G_2 \cong 1 \). The exact sequence \( 1 \rightarrow G_2^{ab} \rightarrow G_2 \rightarrow \tau(G_2, G_2) \rightarrow 0 \) shows that \( \tau(G_2, G_2) \) divides \( p \) where \([G_2^{ab}, G_2] \leq \tau(G_2, G_2) \) and \( \tau(G_2^{ab}, G_2) \leq \tau(G_2^{ab}, G_2) \). Hence \( [G_2^{ab}, G_2] \) divides \( p \). Then from the exact sequence, we obtain \( \tau(G_2, G_2) \cong (\mathbb{Z}_p)^9 \). Since \( \tau(G_2, G_2) \) is abelian and \( \lambda : \tau(G_2, G_2) \rightarrow G_2 \) is the homomorphism, it follows \( p \) divides \( \tau(G_2, G_2) \). Therefore \( G_2 \otimes G_2 \cong (\mathbb{Z}_p)^9 \).

Next, we consider the third case \( G_3 \), by choosing \( n = p^3 \), \( m = p^2 \), \( l = 1 + p \), \( G_3 \) is metacyclic group. By using the same proof as \( G_1 \), then \( G_3 \otimes G_3 = (\mathbb{Z}_p)^2 \times (\mathbb{Z}_p)^2 \).

For group \( G_4 = M_p \times \langle w \rangle \), where \( M_p \) is isomorphic to non-abelian group of order \( p^3 \) of exponent \( p \). Then by Theorem 5, \( G_4 \otimes G_4 \cong (M_p \times \langle w \rangle) \otimes (M_p \times \langle w \rangle) \cong (\mathbb{Z}_p)^{11} \).

For group \( G_5 \), we have \( G_5 = K \times \langle z \rangle \) where \( K = \langle x, y | x^{p^2} = y^p = 1, x^y = x^{1+p} \rangle \). We know that \( K \) is isomorphic to non-abelian group of order \( p^3 \) of exponent \( p^2 \). Again by using Theorem 5, we have \( G_5 \otimes G_5 \cong (K \times \langle z \rangle) \otimes (K \times \langle z \rangle) \cong (\mathbb{Z}_p)^9 \).

Lastly, for group \( G_6 \), we have \( G_6^{ab} = \mathbb{Z}_p \) and \( G_6^{ab} = \mathbb{Z}_p \times \mathbb{Z}_p \). The exact sequence
\[1 \rightarrow [G'_6, G''_6] \rightarrow \tau(G_6, G_6) \rightarrow \tau(G''_6, G''_6) \rightarrow 1.\]

On the other hand, \((G_6 \wedge G_6)/M(G_6) \cong G'_6\). By Theorem 3, \(M(G_6) \cong (\mathbb{Z}_p)^2\), that is \((G_6 \wedge G_6) \cong (\mathbb{Z}_p)^3\) and by Theorem 6, we have \(\Gamma(G''_6) = \Gamma((\mathbb{Z}_p \times \mathbb{Z}_p)^2) = (\mathbb{Z}_p)^2 \times \mathbb{Z}_p^2\). \(G''_6\) is a finitely generated abelian group with no element of order 2. Then by Theorem 7, we have \(G_6 \otimes G_6 \cong \Gamma(G''_6) \times G_6 \wedge G_6 \cong \mathbb{Z}_p^2 \times (\mathbb{Z}_p)^3.\)

\[\square\]

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