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Stability of an alternative Jensen's functional equation

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ABSTRACT: We prove the Hyers-Ulam stability of an alternative Jensen's functional equation $(f(x)+f(y))/2 = \pm f((x+y)/2)$ in the class of mappings from 2-divisible abelian groups to Banach spaces.

KEYWORDS: alternative functional equation, Hyers-Ulam stability, additive function

INTRODUCTION

The problem of the equivalence of the alternative Cauchy functional equation

$$(f(x+y))^2 = (f(x)+f(y))^2 \tag{1}$$

and the Cauchy functional equation

$$f(x+y) = f(x)+f(y) \tag{2}$$

dates back to the work of Hosszú and the work of Vincze referenced in a publication by Kuczma[1]. There are also a few variations of (1), for instance, a more general alternative equation

$$\big(cf(x+y)-af(x)-bf(y)-d\big)\times \big(f(x+y)-f(x)-f(y)\big)=0,$$

which has completely been solved by Forti[2]. Another remarkable result on the alternative Cauchy functional equation

$$f(xy)-f(x)-f(y)\in\{0,1\}$$

where f is a function from a group or a semigroup to $\mathbb{R}$ was recently published in Ref. 3.

Another famous equation that is closely related to the Cauchy functional equation is Jensen's functional equation

$$\frac{1}{2}(f(x)+f(y)) = f\left(\frac{x+y}{2}\right). \tag{3}$$

The solution of (3) on groups can be found in the papers by Ng[4] or Parnami[5]. Similarly to the problem of equivalence of (1) and (2), the author[6] has previously solved the *alternative* Jensen's functional equation

$$\frac{1}{2}(f(x)+f(y)) = \pm f\left(\frac{x+y}{2}\right) \tag{4}$$

on semigroups. But a stability problem of (4) has not yet been investigated.

In this paper, we will prove the Hyers-Ulam stability (cf. Hyers[7], Aoki[8], Bourgin[9], Rassias[10] and Gavruta[11]) of (4) for the class of mappings f from a 2-divisible abelian group $(G,+)$ to a Banach space $(E,\|\cdot\|)$. Namely, for every $\varepsilon \geqslant 0$, we will prove that there exist $\delta^+,\delta^- \geqslant 0$ such that for a mapping $f: G\to E$ satisfying the *alternative* inequalities

$$\left\|\frac{1}{2}(f(x)+f(y)) + f\left(\frac{x+y}{2}\right)\right\| \leqslant \delta^+ \tag{5}$$

$$\text{or}\quad \left\|\frac{1}{2}(f(x)+f(y)) - f\left(\frac{x+y}{2}\right)\right\| \leqslant \delta^-, \tag{6}$$

for every $x,y\in G$, there exists a unique Jensen's mapping $J: G\to E$ satisfying (3) with $J(0)=f(0)$ such that

$$\|f(x)-J(x)\|\leqslant\varepsilon$$

for every $x\in G$. For some previous results on the stability of Jensen's functional equation, readers may consult, for example, Kominek[12], Jung[13], Faĭziev[14], and Kenary[15].

In the subsequent sections, we will start with a derivation of lemmas that bound Jensen's differences (5) and (6) when some alternatives have been decided. All those lemmas will compose another important lemma that will eventually establishes the equivalence of (3) and (4), as well as the Hyers-Ulam stability of the alternative Jensen's functional (4).

AUXILIARY LEMMAS

Throughout the paper, we will consider the class of mappings from a 2-divisible abelian group $(G,+)$ to a Banach space $(E,\|\cdot\|)$. For convenience, we will denote Jensen's differences (5) and (6) of a mapping

$f : G \to E$ by

$$D_f^+(x,y) = \frac{1}{2}(f(x)+f(y)) + f\left(\frac{x+y}{2}\right), \quad (7)$$

$$D_f^-(x,y) = \frac{1}{2}(f(x)+f(y)) - f\left(\frac{x+y}{2}\right). \quad (8)$$

We will let $\delta^+ \geqslant 0$ and $\delta^- \geqslant 0$ be the bounds of (7) and (8), respectively, that is

$$\|D_f^+(x,y)\| \leqslant \delta^+ \quad \text{or} \quad \|D_f^-(x,y)\| \leqslant \delta^- \quad (9)$$

for every $x, y \in G$.

One way to make the Hyers-Ulam stability of (4) feasible is to attempt to bound $D_f^-(x,y)$ for every $x, y \in G$. We can readily see that the alternative $\|D_f^-(x,y)\| \leqslant \delta^-$ in (9) poses no difficulty, while the alternative $\|D_f^+(x,y)\| \leqslant \delta^+$ becomes a real challenge. All lemmas in this section take the assumption $\|D_f^+(x-2y, x+2y)\| \leqslant \delta^-$ and will attempt to bound $D_f^-(x-2y, x+2y)$ by making meaningful observations of f at the points $x-2y, x-y, x, x+y, x+2y$. Consideration of all possible alternatives as in (9) will generally draw sufficient relationship for the desired bound, but in some other cases, further information at the points $x-3y$ and $x+3y$ will play a crucial role towards the determination of the bound.

We will start with the first two lemmas where meaningful observation at the points $x-2y$, $x-y$, $x, x+y, x+2y$ suffices.

Lemma 1 *Suppose a mapping $f : G \to E$ satisfies (9). For every $x, y \in G$, if*

$$\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+,$$
$$\|D_f^+(x-2y, x)\| \leqslant \delta^+,$$
$$\|D_f^+(x, x+2y)\| \leqslant \delta^+,$$

then $\|D_f^-(x-2y, x+2y)\| \leqslant 4\delta^+ + 2\delta^-$.

Proof: Assume the assumptions in the lemma. We consider the following two cases.

(i) If $\|D_f^+(x-y, x+y)\| \leqslant \delta^+$, then we observe that

$$D_f^-(x-2y, x+2y) = D_f^+(x-2y, x) + D_f^+(x, x+2y) - 2D_f^+(x-y, x+y).$$

Hence $\|D_f^-(x-2y, x+2y)\| \leqslant 4\delta^+$.

(ii) If $\|D_f^-(x-y, x+y)\| \leqslant \delta^-$, then we observe that

$$D_f^-(x-2y, x+2y) = 2D_f^-(x-y, x+y) + 2D_f^+(x-2y, x+2y) - D_f^+(x-2y, x) - D_f^+(x, x+2y).$$

Hence $\|D_f^-(x-2y, x+2y)\| \leqslant 4\delta^+ + 2\delta^-$.

The desired bound follows from the consideration of all cases. □

Lemma 2 *Suppose a mapping $f : G \to E$ satisfies (9). For every $x, y \in G$, if*

$$\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+,$$
$$\|D_f^-(x-2y, x)\| \leqslant \delta^-,$$
$$\|D_f^-(x, x+2y)\| \leqslant \delta^-,$$

then $\|D_f^-(x-2y, x+2y)\| \leqslant \max\{4\delta^+ + 2\delta^-,\ 4\delta^-\}$.

Proof: Assume the assumptions in the lemma. We consider the following two cases.

(i) If $\|D_f^+(x-y, x+y)\| \leqslant \delta^+$, then we observe that

$$D_f^-(x-2y, x+2y) = 2D_f^+(x-2y, x+2y) - 2D_f^+(x-y, x+y) - D_f^-(x-2y, x) - D_f^-(x, x+2y).$$

Hence $\|D_f^-(x-2y, x+2y)\| \leqslant 4\delta^+ + 2\delta^-$.

(ii) If $\|D_f^-(x-y, x+y)\| \leqslant \delta^-$, then we observe that

$$D_f^-(x-2y, x+2y) = D_f^-(x-2y, x) + D_f^-(x, x+2y) + 2D_f^-(x-y, x+y).$$

Hence $\|D_f^-(x-2y, x+2y)\| \leqslant 4\delta^-$.

The desired bound follows from the consideration of all cases. □

The next lemma will be considerably more involved as the consideration at the points $x-2y, x-y, x, x+y, x+2y$ is insufficient and thus necessitate further consideration at the point $x+3y$.

Lemma 3 *Suppose a mapping $f : G \to E$ satisfies (9). For every $x, y \in G$, if*

$$\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+,$$
$$\|D_f^+(x-2y, x)\| \leqslant \delta^+,$$
$$\|D_f^-(x, x+2y)\| \leqslant \delta^-,$$
$$\|D_f^+(x-y, x+y)\| \leqslant \delta^+,$$

then

$$\|D_f^-(x-2y, x+2y)\| \leqslant \max\{4\delta^+ + \delta^-, 2\delta^+ + 4\delta^-\}.$$

Proof: Assume the assumptions in the lemma. We consider the following four cases.

(i) If $\|D_f^+(x+y,x+3y)\| \leqslant \delta^+$ and $\|D_f^+(x-y,x+3y)\| \leqslant \delta^+$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= D_f^+(x-2y,x) \\ &- D_f^-(x,x+2y) - D_f^+(x-y,x+y) \\ &+ D_f^+(x+y,x+3y) - D_f^+(x-y,x+3y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 4\delta^+ + \delta^-$.

(ii) If $\|D_f^+(x+y,x+3y)\| \leqslant \delta^+$ and $\|D_f^-(x-y,x+3y)\| \leqslant \delta^-$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= D_f^+(x-2y,x) \\ &- \tfrac{3}{2}D_f^+(x-y,x+y) + \tfrac{1}{2}D_f^+(x+y,x+3y) \\ &- \tfrac{1}{2}D_f^-(x-y,x+3y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 3\delta^+ + \frac{1}{2}\delta^-$.

(iii) If $\|D_f^-(x+y,x+3y)\| \leqslant \delta^-$ and $\|D_f^+(x-y,x+3y)\| \leqslant \delta^+$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= \tfrac{1}{2}D_f^+(x-2y,x) \\ &- \tfrac{1}{2}D_f^-(x,x+2y) + \tfrac{1}{2}D_f^+(x-2y,x+2y) \\ &- \tfrac{3}{2}D_f^+(x-y,x+y) - \tfrac{1}{2}D_f^-(x+y,x+3y) \\ &+ \tfrac{1}{2}D_f^+(x-y,x+3y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 3\delta^+ + \delta^-$.

(iv) If $\|D_f^-(x+y,x+3y)\| \leqslant \delta^-$ and $\|D_f^-(x-y,x+3y)\| \leqslant \delta^-$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= D_f^+(x-2y,x+2y) \\ &- 2D_f^-(x,x+2y) - D_f^+(x-y,x+y) \\ &- D_f^-(x+y,x+3y) + D_f^-(x-y,x+3y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 2\delta^+ + 4\delta^-$.

The desired bound follows from the consideration of all cases. □

Lemma 4 *Suppose a mapping $f : G \to E$ satisfies (9). For every $x, y \in G$, if*

$$\begin{aligned} \|D_f^+(x-2y,x+2y)\| &\leqslant \delta^+, \\ \|D_f^-(x-2y,x)\| &\leqslant \delta^-, \\ \|D_f^+(x,x+2y)\| &\leqslant \delta^+, \\ \|D_f^+(x-y,x+y)\| &\leqslant \delta^+, \end{aligned}$$

then

$$\|D_f^-(x-2y,x+2y)\| \leqslant \max\{4\delta^+ + \delta^-, 2\delta^+ + 4\delta^-\}.$$

Proof: Switching the sign of y in Lemma 3, we immediately get the desired bound. □

The following lemma will resolve the bound of $D_f^-(x-2y,x+2y)$ by an approach similar to that in Lemma 3, but the point $x-3y$ will be considered in lieu of $x+3y$.

Lemma 5 *Suppose a mapping $f : G \to E$ satisfies (9). For every $x, y \in G$, if*

$$\begin{aligned} \|D_f^+(x-2y,x+2y)\| &\leqslant \delta^+, \\ \|D_f^+(x-2y,x)\| &\leqslant \delta^+, \\ \|D_f^-(x,x+2y)\| &\leqslant \delta^-, \\ \|D_f^-(x-y,x+y)\| &\leqslant \delta^-, \end{aligned}$$

then

$$\|D_f^-(x-2y,x+2y)\| \leqslant \max\{4\delta^+ + 2\delta^-,\ 2\delta^+ + 3\delta^-,\ \tfrac{7}{2}\delta^-\}.$$

Proof: Assume the assumptions in the lemma. We consider the following four cases.

(i) If $\|D_f^+(x-3y,x-y)\| \leqslant \delta^+$ and $\|D_f^+(x-3y,x+y)\| \leqslant \delta^+$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= \tfrac{1}{2}D_f^-(x,x+2y) \\ &- \tfrac{1}{2}D_f^+(x-2y,x) + \tfrac{1}{2}D_f^+(x-2y,x+2y) \\ &+ \tfrac{3}{2}D_f^-(x-y,x+y) + \tfrac{1}{2}D_f^+(x-3y,x-y) \\ &- \tfrac{1}{2}D_f^+(x-3y,x+y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 2\delta^+ + 2\delta^-$.

(ii) If $\|D_f^+(x-3y,x-y)\| \leqslant \delta^+$ and $\|D_f^-(x-3y,x+y)\| \leqslant \delta^-$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= D_f^+(x-2y,x+2y) \\ &- 2D_f^+(x-2y,x) + D_f^-(x-y,x+y) \\ &+ D_f^+(x-3y,x-y) - D_f^-(x-3y,x+y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 4\delta^+ + 2\delta^-$.

(iii) If $\|D_f^-(x-3y,x-y)\| \leqslant \delta^-$ and $\|D_f^+(x-3y,x+y)\| \leqslant \delta^+$, then we observe that

$$\begin{aligned} D_f^-(x-2y,x+2y) &= D_f^-(x,x+2y) \\ &- D_f^+(x-2y,x) + D_f^-(x-y,x+y) \\ &- D_f^-(x-3y,x-y) + D_f^+(x-3y,x+y). \end{aligned}$$

Hence $\|D_f^-(x-2y,x+2y)\| \leqslant 2\delta^+ + 3\delta^-$.

(iv) If $\|D_f^-(x-3y, x-y)\| \leqslant \delta^-$ and $\|D_f^-(x-3y, x+y)\| \leqslant \delta^-$, then we observe that

$$D_f^-(x-2y, x+2y) = D_f^-(x, x+2y) + \tfrac{3}{2}D_f^-(x-y, x+y) - \tfrac{1}{2}D_f^-(x-3y, x-y) + \tfrac{1}{2}D_f^-(x-3y, x+y).$$

Hence $\|D_f^-(x-2y, x+2y)\| \leqslant \frac{7}{2}\delta^-$.

The desired bound follows from the consideration of all cases. □

Lemma 6 *Suppose a mapping $f : G \to E$ satisfies* (9). *For every $x, y \in G$, if*

$$\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+,$$
$$\|D_f^-(x-2y, x)\| \leqslant \delta^-,$$
$$\|D_f^+(x, x+2y)\| \leqslant \delta^+,$$
$$\|D_f^-(x-y, x+y)\| \leqslant \delta^-,$$

then

$$\|D_f^-(x-2y, x+2y)\| \leqslant \max\{4\delta^+ + 2\delta^-,\ 2\delta^+ + 3\delta^-,\ \tfrac{7}{2}\delta^-\}.$$

Proof: Switching the sign of y in Lemma 5, we immediately get the desired bound. □

HYERS-ULAM STABILITY

In this section, we aim to prove the Hyers-Ulam stability of the alternative Jensen's functional (4). Firstly, we will put together Lemmas 1–6 in the previous section to conclude the bound of $D^- f(x, y)$.

Lemma 7 *Suppose a mapping $f : G \to E$ satisfies* (9). *For every $x, y \in G$,*

$$\|D_f^-(x, y)\| \leqslant 2\max\{2\delta^+ + \delta^-, \delta^+ + 2\delta^-\}.$$

Proof: For every $x, y \in G$, we know from (9) that

$$\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+ \quad \text{or} \quad \|D_f^-(x-2y, x+2y)\| \leqslant \delta^-.$$

If $\|D_f^+(x-2y, x+2y)\| \leqslant \delta^+$, then Lemmas 1-6 give

$$\|D_f^-(x-2y, x+2y)\| \leqslant 2\max\{2\delta^+ + \delta^-,\ \delta^+ + 2\delta^-\}. \quad (10)$$

Otherwise, $\|D_f^-(x-2y, x+2y)\| \leqslant \delta^-$, which readily satisfies (10). Hence, (10) holds for every $x, y \in G$. Replacing x with $(x+y)/2$ and y with $(y-x)/4$ in (10), we get the desired result. □

As a direct consequence of Lemma 7, if we restrict the values of both δ^+ and δ^- in (9) to zero, then $\|D^- f(x, y)\|$ will be confined to zero as well. The following theorem employs this fact to establish the equivalence of Jensen's functional (3) and the alternative Jensen's functional (4).

Theorem 1 *A mapping $f : G \to E$ satisfies* (4) *if and only if it satisfies* (3).

Proof: If f satisfies (4), then setting $\delta^+ = \delta^- = 0$ in Lemma 7 gives $D_f^-(x, y) = 0$ for every $x, y \in G$; that is, f satisfies (3). Conversely, if f satisfies (3), then f readily satisfies (4). □

It should be remarked here that the 2-divisibility (or possibly other substitutes) of the group $(G, +)$ in Theorem 1 is crucial. The 2-divisibility of $(G, +)$ seems natural in (3), but if we relax Jensen's functional equation to

$$\tfrac{1}{2}(f(2x) + f(2y)) = f(x+y), \quad (11)$$

and accordingly relax the alternative Jensen's functional (4) to

$$\tfrac{1}{2}(f(2x) + f(2y)) = \pm f(x+y), \quad (12)$$

then we can give an example of a mapping which satisfies (12) but does not satisfy (11).

Example 1 Consider the addition group of $\mathbb{Z}$. Define a mapping $f : \mathbb{Z} \to \mathbb{R}$ by $f(n) = (-1)^n$ for every $n \in \mathbb{Z}$. For every $m, n \in \mathbb{Z}$, $f(2m) + f(2n) = 2$. Hence f satisfies (12). But $f(1) \neq \frac{1}{2}(f(0) + f(2))$. Hence, f does not satisfy (11).

The following theorem presents the Hyers-Ulam stability of the alternative Jensen's functional (4) in the class of mappings for 2-divisible abelian groups to Banach spaces using the so-called *direct* method. For the stability results of Jensen's functional equation, please refer to, for instance, Kominek[12] or Jung[13].

Theorem 2 *If a mapping $f : G \to E$ satisfies* (9), *then there exists a Jensen's mapping $J : G \to E$ satisfying* (3) *with $J(0) = f(0)$ such that*

$$\|f(x) - J(x)\| \leqslant 4\max\{2\delta^+ + \delta^-, \delta^+ + 2\delta^-\}$$

for every $x \in G$. The mapping J is given by

$$J(x) = f(0) + \lim_{n\to\infty} \frac{1}{2^n}(f(2^n x) - f(0))$$

for every $x \in G$.

Proof: Suppose a mapping $f : G \to E$ satisfies (9). We define another mapping $\tilde{f} : G \to E$ by

$$\tilde{f}(x) = f(x) - f(0) \quad \text{for every } x \in G.$$

It should be noted that $D^-_{\tilde{f}}(x,y) = D^-_f(x,y)$.

Let $\delta = 2\max\{2\delta^+ + \delta^-, \delta^+ + 2\delta^-\}$. For every $x, y \in G$, Lemma 7 yields

$$\|D^-_{\tilde{f}}(x,y)\| \leqslant \delta,$$

that is

$$\left\|\frac{1}{2}(\tilde{f}(x) + \tilde{f}(y)) - \tilde{f}\left(\frac{x+y}{2}\right)\right\| \leqslant \delta. \tag{13}$$

Setting $(x,y) := (2x, 0)$ in (13) and knowing that $\tilde{f}(0) = 0$, we have

$$\left\|\frac{\tilde{f}(2x)}{2} - \tilde{f}(x)\right\| \leqslant \delta$$

for every $x \in G$. With n being a positive integer, we observe that

$$\frac{\tilde{f}(2^n x)}{2^n} - \tilde{f}(x) = \sum_{i=1}^{n}\left(\frac{\tilde{f}(2^i x)}{2^i} - \frac{\tilde{f}(2^{i-1}x)}{2^{i-1}}\right).$$

Hence

$$\left\|\frac{\tilde{f}(2^n x)}{2^n} - \tilde{f}(x)\right\| \leqslant \delta\left(2 - \frac{1}{2^{n-1}}\right) \tag{14}$$

for every $x \in G$ and for every positive integer n.

Consider the sequence $\{2^{-n}\tilde{f}(2^n x)\}$. As a result of (14), we have

$$\begin{aligned}
&\left\|\frac{\tilde{f}(2^{n+m}x)}{2^{n+m}} - \frac{\tilde{f}(2^n x)}{2^n}\right\| \\
&\quad = \frac{1}{2^n}\left\|\frac{\tilde{f}(2^m \cdot 2^n x)}{2^m} - \tilde{f}(2^n x)\right\| \\
&\quad \leqslant \frac{\delta}{2^n}\left(2 - \frac{1}{2^{m-1}}\right).
\end{aligned}$$

Hence we can see that $\{2^{-n}\tilde{f}(2^n x)\}$ is a Cauchy sequence. It is now legitimate to define a mapping $\tilde{J} : G \to E$ by

$$\tilde{J}(x) = \lim_{n\to\infty}\frac{\tilde{f}(2^n x)}{2^n}$$

for every $x \in G$. Replacing x and y in (13) with $2^n x$ and $2^n y$, respectively, then taking the limit as $n \to \infty$, we can see that J is a Jensen's mapping satisfying (3). Considering (14), as $n \to \infty$, we obtain, for every $x \in G$,

$$\|\tilde{f}(x) - \tilde{J}(x)\| \leqslant 2\delta.$$

The uniqueness of $\tilde{J}$ can be shown by assuming an existence of another Jensen's mapping $\tilde{J}' : G \to E$ such that $\tilde{J}'(0) = 0$ and $\|\tilde{f}(x) - \tilde{J}'(x)\| \leqslant 2\delta$ for every $x \in G$. With a positive integer n, we have $\tilde{J}(2^n x) = 2^n\tilde{J}(x)$ and $\tilde{J}'(2^n x) = 2^n\tilde{J}'(x)$. Hence

$$\begin{aligned}
&\|\tilde{J}(x) - \tilde{J}'(x)\| \\
&\quad = \left\|\frac{\tilde{f}(2^n x) - \tilde{J}'(2^n x)}{2^n} - \frac{\tilde{f}(2^n x) - \tilde{J}(2^n x)}{2^n}\right\| \\
&\quad \leqslant \frac{1}{2^n}\left\|\tilde{f}(2^n x) - \tilde{J}'(2^n x)\right\| \\
&\qquad + \frac{1}{2^n}\left\|\tilde{f}(2^n x) - \tilde{J}(2^n x)\right\| \\
&\quad \leqslant \frac{4\delta}{2^n}.
\end{aligned}$$

As $n \to \infty$, we can see that $\tilde{J}(x) = \tilde{J}'(x)$ for every $x \in G$.

We conclude the theorem by defining a mapping $J : G \to E$ by $J(x) = \tilde{J}(x) + f(0)$ for every $x \in G$ to give all desired results. □

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