A characterization of \( L_3(4) \)

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Abstract: Let \( G \) be a group and \( \omega(G) \) be the set of element orders of \( G \). Let \( k \in \omega(G) \) and \( s_k \) be the number of elements of order \( k \) in \( G \). Let \( nse(G) = \{ s_k \mid k \in \omega(G) \} \). \( L_3(2) \cong L_2(7) \) is uniquely determined by \( nse(G) \). In this paper, we prove that if \( G \) is a group such that \( nse(G) = nse(L_3(4)) \), then \( G \cong L_3(4) \).

Keywords: element order, linear group, Thompson’s problem, number of elements of the same order

Introduction

If \( n \) is an integer, then we denote by \( \pi(n) \) the set of all prime divisors of \( n \). Let \( G \) be a group. The set of element orders of \( G \) is denoted by \( \omega(G) \). Let \( k \in \omega(G) \) and \( s_k \) be the number of elements of order \( k \) in \( G \). Let \( nse(G) = \{ s_k \mid k \in \omega(G) \} \). Let \( \pi(G) \) denote the set of primes \( p \) such that \( G \) contains an element of order \( p \). A finite group \( G \) is called a simple \( K_n \)-group if \( G \) is a simple group with \( |\pi(G)| = n \). Thompson posed a very interesting problem related to algebraic number fields as follows.

Thompson’s Problem. Let \( T(G) = \{ (n, s_n) \mid n \in \omega(G) \text{ and } s_n \in nse(G) \} \), where \( s_n \) is the number of elements of order \( n \). Suppose that \( T(G) = T(H) \). If \( G \) is a finite solvable group, is it true that \( H \) is also necessarily solvable?

It was proved that if \( G \) is a group and \( M \) some simple \( K_3 \)-group, \( i = 3, 4 \), then \( G \cong M \) if and only if \( |G| = |M| \) and \( nse(G) = nse(M) \) (see Refs. 2, 3). And the groups \( A_{12}, A_{13} \) and \( L_3(2) \) are characterizable by \( nse \) (see Refs. 4–6).

We only consider the sizes of elements of the same order but disregard the actual orders of elements in \( T(G) \) of Thompson’s Problem. In other words, can \( nse(G) \) characterize finite simple groups? Some groups for \( L_2(q) \), where \( q \in \{ 7, 8, 9, 11, 13 \} \), have been characterized by only the set \( nse(G) \) (see Refs. 7, 8). In this paper it is shown that the projective special linear group \( L_3(4) \) can also be characterized by \( nse(L_3(4)) \).

Some Lemmas

Lemma 1 (Ref. 9) Let \( G \) be a finite group and \( m \) be a positive integer dividing \( |G| \). If \( L_m(G) = \{ g \in G \mid g^m = 1 \} \), then \( m \mid |L_m(G)| \).

Lemma 2 (Ref. 10) Let \( G \) be a finite group and \( p \in \pi(G) \) be odd. Suppose that \( P \) is a Sylow \( p \)-subgroup of \( G \) and \( n = p^s m \) with \( (p, m) = 1 \). If \( P \) is not cyclic and \( s > 1 \), then the number of elements of order \( n \) is always a multiple of \( p^s \).

Lemma 3 (Ref. 8) Let \( G \) be a group containing more than two elements. If the maximum number \( s \) of elements of the same order in \( G \) is finite, then \( G \) is finite and \( |G| \leq s^2 - 1 \).

Lemma 4 (Theorem 9.3.1 of Ref. 11) Let \( G \) be a finite soluble group and \( |G| = mn \), where \( m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \) and \( (m, n) = 1 \). Let \( \pi = \{ p_1, \ldots, p_r \} \) and \( h_m \) be the number of Hall \( \pi \)-subgroups of \( G \). Then \( h_m = q_1^{\beta_1} \cdots q_s^{\beta_s} \) satisfies the following conditions for all \( i \in \{ 1, 2, \ldots, s \} \):

(i) \( q_i^{\beta_i} \equiv 1 \pmod{p_j} \) for some \( p_j \).

(ii) The order of some chief factor of \( G \) is divisible by \( q_i^{\beta_i} \).

Proof of Theorem

Let \( G \) be a group such that \( nse(G) = nse(L_3(4)) \), and \( s_n \) be the number of elements of order \( n \). By Lemma 3 we have \( G \) is finite. We note that \( s_n = k\phi(n) \), where \( k \) is the number of cyclic subgroups of order \( n \). Also we note that if \( n > 2 \), then \( \phi(n) \) is even. If \( m \in \omega(G) \), then by Lemma 1 and the above discussion, we have

\[
\begin{align*}
\phi(m) | s_m, \\
m | \sum_d |m, s_d.
\end{align*}
\]

Theorem 1 Let \( G \) be a group with \( nse(G) \) nse\((L_3(4)) = \{ 1, 315, 2240, 3780, 5760, 8064 \} \), where \( L_3(4) \) is the projective special linear group of degree 3 over the finite field of order 4. Then \( G \cong L_3(4) \).

Proof: We prove the theorem by first proving that \( \pi(G) \subseteq \{ 2, 3, 5, 7 \} \), second showing that \( |G| = |L_3(4)| \), and so \( G \cong L_3(4) \).
By (i), \( \pi(G) \subseteq \{2, 3, 5, 7, 19\} \). If \( m > 2 \), then \( \phi(m) \) is even, then \( s_2 = 315, 2 \in \pi(G) \). If \( 3, 5, 7 \in \pi(G) \), then \( s_3 = 2240, s_5 = 8064 \), and \( s_7 = 5760 \). In the following, we prove that \( 19 \notin \pi(G) \). If \( 19 \in \pi(G) \), then by (i), \( s_{19} = 3780 \). If \( 2 \cdot 19 \in \omega(G) \), then \( s_{38} \notin \nse(G) \). Therefore \( 38 \notin \omega(G) \). It follows that the Sylow 19-subgroup \( P_{19} \) acts fixed point freely on the set of elements of order 2, then \( |P_{19}| \) is not \( 315 \), a contradiction. Hence \( \pi(G) \subseteq \{2, 3, 5, 7\} \).

If \( 2^i \in \omega(G) \), then \( \phi(2^i) = 2^{i-1} \cdot s_2 \), and so \( 1 \leq i \leq 8 \). If \( 3^j \in \omega(G) \), then \( \phi(3^j) \mid s_3 \), and so \( 1 \leq j \leq 4 \). If \( 5^k \in \omega(G) \), then \( 1 \leq k \leq 2 \). If \( 5^2 \in \omega(G) \), then \( s_{25} \notin \nse(G) \), a contradiction. Hence \( k = 1 \).

1. If \( 2^l \in \omega(G) \), then \( 1 \leq l \leq 2 \). If \( 2^2 \in \omega(G) \), then \( s_{49} \notin \nse(G) \), a contradiction. Therefore \( l = 1 \).

If \( 2^n \cdot 3^m \in \omega(G) \), then \( 1 \leq m \leq 7 \) and \( 1 \leq n \leq 4 \). If \( 2^n \cdot 5 \in \omega(G) \), then \( 1 \leq a \leq 6 \). If \( 2^b \cdot 7 \in \omega(G) \), then \( 1 \leq b \leq 7 \).

2. If \( 3^c \cdot 5 \in \omega(G) \), then \( 1 \leq c \leq 3 \). By (i), \( s_{15} = s_{45} = 5760 \). If \( 3^d \cdot 7 \in \omega(G) \), then \( 1 \leq d \leq 3 \).

3. If \( 5^f \in \omega(G) \), the \( s_{25} \notin \nse(G) \), a contradiction.

Hence \( 5 \cdot 7 \notin \omega(G) \).

Case (b). \( \pi(G) = \{2, 3\} \). We know that \( \exp(P_3) = 3, 9, 27 \), or 81. By (i), \( s_9, s_{27} \in \{3780, 5760, 8064\} \) and so we get that \( s_{81} \notin \nse(G) \). Hence \( \exp(P_3) = 3, 9, 27 \).

Subcase b.1. \( \exp(P_3) = 3 \). By Lemma 1, \( |P_3| \mid 1 + s_3 \), and so \( |P_3| \leq 9 \). If \( |P_3| = 3 \), then \( P_3 \) is cyclic and by (i), \( n_3 = s_3/\phi(3) = 2240/2 = 1120 \), which means \( 5 \mid |G| \), a contradiction. If \( |P_3| = 9 \), then \( 5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{4-2} \cdot 3^2 \cdot 7^2 \cdot 17 \), where \( k_1, k_2, k_3, k_4 \) and \( m \) are non-negative integers and \( 0 \leq k_1 + k_2 + k_3 + k_4 \leq 11 \). Hence we have \( 5040 \leq 2^{4-2} \cdot 3^2 \leq 5040 \cdot 2016 \cdot 11 \) and so \( l = 10 \) or 11.

Case \( l = 10 \). Therefore \( 9(560 + 105k_2 + 224k_4) + 560k_1 + 1440k_3 \in 2^{6-2} \cdot 3^2 \). Thus \( 9 \mid k_1 \), and so \( k_1 = 0 \) or \( k_1 = 9 \).

Case \( k_3 = 0 \). If \( k_3 = 0 \), then \( 16(35 + 14k_3) + 105k_2 = 2^{10} \) and so \( k_2 = 0 \). But the equation \( 35 + 14k_3 \) is \( 2^6 \) has no solution. If \( k_3 = 3 \), then \( 4(140 + 95k_3 + 6k_4) + 105k_2 = 2^{10} \), and so \( 4 \mid k_2 \). Hence \( k_2 = 0, 4, 8 \). If \( k_2 = 0 \), then the equation \( 560 + 480 + 224k_4 = 2^{10} \) has no solution in \( \mathbb{N} \). If \( k_4 = 2 \), then the equation \( 1360 + 224k_4 = 1024 \) has no solution in \( \mathbb{N} \). Then \( k_2 = 8 \) and so \( k_4 = 0 \), so we also get a contradiction. If \( k_3 = 9 \), then \( 4 \mid k_2 \) and so \( k_2 = 0 \). Therefore we have that \( 1700 + 224k_4 = 2^{10} \), but the equation has no solution in \( \mathbb{N} \).

Case \( k_1 = 9 \). We can get the same result as \( k_1 = 0 \).

Case \( l = 11 \). We also get same the results as \( l = 10 \).

Subcase b.2. \( \exp(P_3) = 9 \). By Lemma 1, \( |P_3| \mid 1 + s_3 + s_9 \) and so \( |P_3| \leq 27 \). If \( |P_3| = 9 \), then \( P_3 \) is cyclic. By (i), \( s_9 = 3780, 5760, 8064 \), and so \( n_3 = s_9/\phi(9) = 945, 1440, 2016 \). It follows that \( 5 \mid |G| \) or \( 7 \mid |G| \), a contradiction. If \( |P_3| = 27 \), then \( 5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{4-2} \cdot 3^2 \cdot 7 \), where \( k_1, k_2, k_3, k_4, m \) are non-negative integers and \( 0 \leq k_1 + k_2 + k_3 + k_4 \leq 16 \). Hence we have \( 5040 \leq 2^{4-2} \cdot 3^2 \leq 5040 \cdot 2016 \cdot 16 \) and so \( l = 8, 9 \), or 10.

Case \( l = 8 \). We have that \( 9 \mid k_1 \), and so \( k_1 = 0 \) or \( 9 \). If \( k_1 = 0 \), then \( 16(35 + 10k_3 + 14k_4) + 105k_2 = 2^6 \cdot 3 \), it follows that \( 16 \mid k_2 \). Hence \( k_2 = 0 \), and so \( 35 + 10k_3 + 14k_4 = 12 \), but the equation has no solution in \( \mathbb{N} \).

Case \( l = 9, 10 \). As with the case \( l = 8 \), we get a contradiction.

Subcase b.3. \( \exp(P_3) = 27 \). By Lemma 1, \( |P_3| \mid 1 + s_3 + s_9 + s_{27} \), and so \( |P_3| \leq 81 \). If \( |P_3| = 27 \), then \( P_3 \) is cyclic. By (i), \( s_{27} = 3780, 5760 \) or 8064 and so \( n_3 = s_{27}/\phi(27) = 210, 320, 448 \), it follows that \( 5 \mid |G| \) or \( 7 \mid |G| \), a contradiction. If \( |P_3| = 81 \),
then by Lemma 2, $s_{81} = 27t$ for some integer $t$ and so $s_{27} = 3780$. But by Lemma 1, $27 | 1 + s_3 + s_9 + s_{27}$ (as 9800, 11780), a contradiction.

Case (e). $\pi(G) = \{ 2, 5 \}$. Since $s_5 = 8064$, by Lemma 1, 5 $|$ 1 $+$ s_5, so we have $|P_5| = 5$. Then by (i) $n_5(G) = s_5/\phi(5) = 2016$, which means that $7 | |G|$, a contradiction.

Case (d). $\pi(G) = \{ 2, 7 \}$. Since $s_7 = 5760$, by Lemma 1, 7 $|$ 1 $+$ s_7, so we have $|P_7| = 7$. Then by (i) $n_7(G) = s_7/\phi(7) = 960$, which means that $5 | |G|$, a contradiction.

Case (e). $\pi(G) = \{ 2, 3, 5 \}$. The proof is similar to Case (c).

Case (f). $\pi(G) = \{ 2, 3, 7 \}$. The proof is similar to Case (d).

Case (g). $\pi(G) = \{ 2, 5, 7 \}$. The proof is similar to Case (c) or (d).

Case (h). $\pi(G) = \{ 2, 3, 5, 7 \}$. In the following, we first show that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ or $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, then prove that there is no group such that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ and nse($G$) = nse($L_3(4)$), and show that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and nse($G$) = nse($L_3(4)$), we have $G \cong L_3(4)$.

Step 1. $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ or $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$

We know that $|P_3| = 5$, $|P_5| = 7$. We will show that $15 \notin \omega(G)$. If $15 \in \omega(G)$, set $P$ and $Q$ are Sylow 5-subgroups of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_{15} = \phi(15) \cdot n_5 \cdot k$, where $k$ is the number of cyclic subgroups of order 3 in $C_G(P_5)$. As $n_3 = s_3/\phi(3) = 8064/4 = 1062$, $|s_{15} = 8064$. But 15 $|$ 1 $+$ s_3 + s_5 + s_{15} = 18369, a contradiction. We conclude that $15 \notin \omega(G)$.

It follows that the group $P_3$ acts fixed point freely on the set of elements of order 19 so $|P_3| | s_5 = 8064$. So we have $|P_5| = 3^2$.

We will show that $14 \notin \omega(G)$. If $14 \in \omega(G)$, set $P$ and $Q$ are Sylow 7-subgroups of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_{14} = \phi(14) \cdot n_7 \cdot k$, where $k$ is the number of cyclic subgroups of order 2 in $C_G(P_7)$. As $n_7 = s_7/\phi(7) = 5760/6 = 960$, $|s_{14} = 5760$. But 14 $|$ 1 $+$ s_2 + s_7 + s_{14} = 11836, we get a contradiction. We have that $14 \notin \omega(G)$. It follows that the group $P_2$ acts fixed point freely on the set of elements of order 7 and so $|P_2| | s_7 = 8064$. So we have $|P_2| = 2^7$. Therefore we have $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. But

$$\sum_{s_{81} \in \text{nse}(G)} s_{81} = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \leq 2^7 \cdot 3^2 \cdot 5 \cdot 7.$$ 

So we have the results.

Step 2. $G \cong L_3(4)$. First show that there is no group such that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ and nse($G$) = nse($L_3(4)$). Then get the result in Ref. 3. Since $s_7 = 5760, n_7 = s_7/\phi(7) = 5760/6 = 2^3 \cdot 5 \cdot 7$ since $G$ is soluble, then by Lemma 4, $5 \equiv 1 \pmod{7}$, a contradiction. So $G$ is insoluble. Therefore we can suppose that $G$ has a normal series $1 \triangleleft K \triangleleft L \triangleleft G$ such that $L/K$ is isomorphic to a simple $K_i$-group with $i = 3, 4$ as 25 and 49 do not divide order of $G$. If $L/K$ is isomorphic to a $K_4$-simple group, then from Ref. 12 $L/K \cong A_5, A_6, L_2(7), L_2(8), U_3(3)$, or $U_4(2)$. From Ref. 13, $n_3(L/K) = n_5(A_5) = 6$, and so $n_5(G) = 6t$ and 5 $|$ t for some integer $t$. Hence the number of elements of order 5 in $G$ is $s_5 = 6t \cdot 4 = 24t$. Since $s_5 \in \text{nse}(G)$, then $s_5 = 8064$ and so $t = 2016$. Therefore $2^5 \cdot 3^2 \cdot 7 | |K| \cdot |2^4 \cdot 3 \cdot 7|$, which is a contradiction. For the other cases, similarly we can rule out these. If $L/K$ is isomorphic to a $K_4$-simple group, then from Ref. 14, we have the following. $L/K$ is isomorphic to one of the following groups: $A_7, A_8, A_9, A_{10}; L_2(49), L_3(4), S_4(7), S_6(2), U_3(5), U_4(3), J_2$, or $O_6(2)$.

If $L/K \cong A_7$, then from Ref. 13, $n_7(L/K) = 120$, and so $n_7(G) = 120t$ with 7 $|$ t. Hence the number of elements of order 7 in $G$ is $s_7 = 120t \cdot 6 = 720t$. Thus $s_7 = 5760$ and $t = 8$. Now $n_7(G) = 960$. On the other hand, by Sylow’s Theorem $n_7(G) = 1, 8, 64$ or 288, a contradiction. For the remaining groups except $L_3(4)$, we can also rule out by the methods as $A_7$.

In the following we show that $G \cong 2L_3(4)$. From above, $L/K \cong L_3(4)$. Let $G = G/K$ and $L = L/K$. Then $L_3(4) \leq L \leq GC_G(L)/C_G(L) \leq G/C_G(L) = N_G(L)/C_G(L) \leq \text{Aut}(L)$.

Set $M = \{ xK \times xK \in C_G(L) \}$, then $G/M \cong GC_G(L)/C_G(L)$ and so $L_3(4) \leq G/M \leq \text{Aut}(L_3(4))$. Therefore $G/M \cong L_3(4)$, $G/M \cong 2L_3(4)$ or $G/M$ is isomorphic to 3$L_3(4)$, $S_3L_3(4)$, $2S_3L_3(4)$, or $2.2L_3(4)$.

If $G/M$ is isomorphic to 3$L_3(4)$, $S_3L_3(4)$, $2S_3L_3(4)$ or 2.2$L_3(4)$, then order consideration rules out this case.

If $G/M \cong L_3(4), |M| = 2$ and so $G$ has a normal subgroup of order 2, which is generated by a central involution. Thus $G$ has an element of order 14, which is a contradiction.

If $G/M \cong 2L_3(4), |M| = 1$. By Sylow’s theorem, $n_7(G) = 1, 8, 36, 64, 288$. On the other hand, since $s_7 = 5760$ and $\exp(P_7) = 7$, we have $n_7 = s_7/\phi(7) = 5760/6 = 960$, a contradiction.

Second, since $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and nse($G$) = nse($L_3(4)$), we have from Ref. 3, $G \cong L_3(4)$. This completes the proof.
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REFERENCES