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Some subsemigroups of the full transformation semigroups

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ABSTRACT: A nonempty subset T of a semigroup S is called a subsemigroup of S if $T^2 \subseteq T$, that is, $xy \in T$ for all $x, y \in T$. For a nonempty set X, let $T(X)$ be the semigroup of the full transformations on X, which consists of all functions from X into X with composition as the semigroup operation. The purpose of this paper is to define various subsemigroups of $T(X)$. Necessary and sufficient conditions are given for these subsemigroups of $T(X)$ to be equal.

KEYWORDS: equivalence relation, totally ordered set, transformation semigroup

INTRODUCTION

Let S be a semigroup. A nonempty subset T of S is called a *subsemigroup* of S if $T^2 \subseteq T$, that is, $xy \in T$ for all $x, y \in T$.

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under composition of mappings.

For a totally ordered set $(X, \leqslant)$, let E be an equivalence relation on X. Higgins[1], Namnak and Laysirikul[2] and Pei[3] have studied the subsemigroups of $T(X)$ as follows.

$$
\begin{aligned}
T_{\mathrm{O}}(X) &= \{\alpha \in T(X) : \forall x, y \in X, x \leqslant y \\
&\qquad \text{implies } x\alpha \leqslant y\alpha\}, \\
T_{\mathrm{R}}(X) &= \{\alpha \in T(X) : \forall x \in X, x\alpha \leqslant x\}, \\
T_{\mathrm{SE}}(X) &= \{\alpha \in T(X) : \forall x \in X, (x, x\alpha) \in E\}, \\
T_{\mathrm{E}}(X) &= \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \\
&\qquad \text{implies } (x\alpha, y\alpha) \in E\}.
\end{aligned}
$$

All subsemigroups clearly contain i_X where i_X is the identity map on X. Next, we define subsets of $T(X)$ by

$$
\begin{aligned}
T_{\mathrm{ER}}(X) &= T_{\mathrm{E}}(X) \cap T_{\mathrm{R}}(X), \\
T_{\mathrm{SER}}(X) &= T_{\mathrm{SE}}(X) \cap T_{\mathrm{R}}(X), \\
T_{\mathrm{EO}}(X) &= T_{\mathrm{E}}(X) \cap T_{\mathrm{O}}(X), \\
T_{\mathrm{SEO}}(X) &= T_{\mathrm{SE}}(X) \cap T_{\mathrm{O}}(X), \\
T_{\mathrm{OR}}(X) &= T_{\mathrm{O}}(X) \cap T_{\mathrm{R}}(X).
\end{aligned}
$$

It is known that the intersection of subsemigroups of a semigroup S is either an empty set or itself a subsemigroup of S. Then $T_{\mathrm{ER}}(X)$, $T_{\mathrm{SER}}(X)$, $T_{\mathrm{EO}}(X)$, $T_{\mathrm{SEO}}(X)$ and $T_{\mathrm{OR}}(X)$ are subsemigroups of $T(X)$ containing i_X.

In this paper the set X under consideration is a totally ordered set with E an arbitrary equivalence relation on X. We denote by X/E the family of E-classes on X and by $|A|$ the cardinality of a set A.

MAIN RESULTS

In this section, we characterize the conditions under which some of above subsemigroups of $T(X)$ are equal.

Theorem 1 *$T_{\mathrm{ER}}(X) = T_{\mathrm{R}}(X)$ if and only if for all $A, B \in X/E$ such that $A \neq B$, if there exist $a \in A$, $b \in B$ such that $a < b$, then $|B| = 1$.*

Proof: Assume that there exist $A, B \in X/E$ such that $A \neq B$, if $a < b$ for some $a \in A$, $b \in B$ and $|B| > 1$. Define $\alpha : X \to X$ by

$$
x\alpha = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise.} \end{cases}
$$

Clearly, $x\alpha \leqslant x$ for all $x \in X$ and then $\alpha \in T_{\mathrm{R}}(X)$. Let $c \in B \backslash \{b\}$. Thus $(b, c) \in E$. Since $(b\alpha, c\alpha) = (a, c) \notin E$, $\alpha \notin T_{\mathrm{E}}(X)$. Therefore $T_{\mathrm{ER}}(X) \neq T_{\mathrm{R}}(X)$.

Conversely, suppose that for all $A, B \in X/E$ such that $A \neq B$, if there exist $a \in A$, $b \in B$ such that $a < b$, then $|B| = 1$. To show that $T_{\mathrm{ER}}(X) = T_{\mathrm{R}}(X)$, let $\alpha \in T_{\mathrm{R}}(X)$ and let $x, y \in X$ be such that $(x, y) \in E$. Hence $x, y \in B$ for some $B \in X/E$. If $x = y$, then $(x\alpha, y\alpha) \in E$. Suppose that $x \neq y$. Then $|B| > 1$. Since $\alpha \in T_{\mathrm{R}}(X)$, $x\alpha \leqslant x$. It follows by assumption that $x\alpha \in B$. Similarly, we have that $y\alpha \in B$. This means that $(x\alpha, y\alpha) \in E$. Therefore, $\alpha \in T_{\mathrm{E}}(X)$ and hence $T_{\mathrm{ER}}(X) = T_{\mathrm{R}}(X)$. □

Theorem 2 $T_{\mathrm{ER}}(X) = T_{\mathrm{E}}(X)$ *if and only if* $|X| = 1$.

Proof: Suppose that $|X| > 1$. Let $a, b \in X$ be such that $a \neq b$. Then there exist $A, B \in X/E$ such that $a \in A$ and $b \in B$. Suppose that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in A, \\ x, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ be such that $(x, y) \in E$. Then

$$(x\alpha, y\alpha) = \begin{cases} (b, b) \in E, & \text{if } x, y \in A, \\ (x, y) \in E, & \text{otherwise} \end{cases}$$

which implies that $\alpha \in T_{\mathrm{E}}(X)$. Since $a\alpha = b \nleqslant a$, we deduce that $\alpha \notin T_{\mathrm{ER}}(X)$. □

Theorem 3 $T_{\mathrm{SER}}(X) = T_{\mathrm{R}}(X)$ *if and only if* $E = X \times X$.

Proof: Suppose that $E \neq X \times X$. Then there exist $a, b \in X$ such that $(a, b) \notin E$. Suppose that $a < b$ and define $\alpha : X \to X$ as given in Theorem 1. Then $\alpha \in T_{\mathrm{R}}(X)$. Since $(b, b\alpha) = (b, a) \notin E$, $\alpha \notin T_{\mathrm{SE}}(X)$. Hence $T_{\mathrm{SER}}(X) \neq T_{\mathrm{R}}(X)$.

Assume that $E = X \times X$. We have that $T_{\mathrm{SE}}(X) = T(X)$, hence $T_{\mathrm{SER}}(X) = T_{\mathrm{SE}}(X) \cap T_{\mathrm{R}}(X) = T(X) \cap T_{\mathrm{R}}(X) = T_{\mathrm{R}}(X)$. □

Theorem 4 $T_{\mathrm{SER}}(X) = T_{\mathrm{SE}}(X)$ *if and only if* $E = I_X$ *where* I_X *is the identity relation on* X.

Proof: Suppose that $E \neq I_X$. Then there exist $a, b \in X$ such that $a \neq b$ and $(a, b) \in E$. We may assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise.} \end{cases}$$

For each $x \in X$,

$$(x, x\alpha) = \begin{cases} (a, b) \in E, & \text{if } x = a, \\ (x, x) \in E, & \text{otherwise,} \end{cases}$$

hence $\alpha \in T_{\mathrm{SE}}(X)$. Since $a\alpha = b \nleqslant a$, we conclude that $\alpha \notin T_{\mathrm{R}}(X)$. Therefore $T_{\mathrm{SER}}(X) \neq T_{\mathrm{SE}}(X)$.

Conversely, suppose that E is the identity relation on X. Let $\alpha \in T_{\mathrm{SE}}(X)$ and $x \in X$. Then we have that $(x, x\alpha) \in E$. By assumption, $x\alpha = x$ which implies that $\alpha \in T_{\mathrm{R}}(X)$. This proves that $T_{\mathrm{SE}}(X) \subseteq T_{\mathrm{R}}(X)$, hence $T_{\mathrm{SE}}(X) = T_{\mathrm{SE}}(X) \cap T_{\mathrm{R}}(X) = T_{\mathrm{SER}}(X)$. □

Theorem 5 $T_{\mathrm{EO}}(X) = T_{\mathrm{O}}(X)$ *if and only if* $E = X \times X$ *or* $E = I_X$.

Proof: Assume that $E \neq X \times X$ and $E \neq I_X$. Then there exist $A, B \in X/E$ such that $|A| > 1$ and $B \neq A$. Let $a, c \in A$ be such that $a < c$ and $b \in B$. Since E is an equivalence relation on X, we have that $b \in B \backslash A$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} \max(b, c), & \text{if } c \leqslant x, \\ \min(b, c), & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_{\mathrm{O}}(X)$, let $x, y \in X$ be such that $x \leqslant y$.

Case 1. If $c \leqslant x \leqslant y$ or $x \leqslant y < c$, then we get that $x\alpha = y\alpha$.

Case 2. If $x < c \leqslant y$, then we have that $x\alpha = \min(b, c) < \max(b, c) = y\alpha$.

From two cases, we deduce that $\alpha \in T_{\mathrm{O}}(X)$. Since $(a, c) \in E$ and $(a\alpha, c\alpha) = (\min(b, c), \max(b, c)) \notin E$, we deduce that $\alpha \notin T_{\mathrm{E}}(X)$. This proves that $T_{\mathrm{EO}}(X) \neq T_{\mathrm{O}}(X)$ as required.

Conversely, suppose that $E = X \times X$ or $E = I_X$. We then have that $T_{\mathrm{E}}(X) = T(X)$. Therefore $T_{\mathrm{EO}}(X) = T_{\mathrm{E}}(X) \cap T_{\mathrm{O}}(X) = T(X) \cap T_{\mathrm{O}}(X) = T_{\mathrm{O}}(X)$. □

Theorem 6 $T_{\mathrm{EO}}(X) = T_{\mathrm{E}}(X)$ *if and only if* $|X| = 1$.

Proof: Assume that $|X| > 1$. Let $a, b \in X$ be such that $a < b$. Then $a \in A$ for some $A \in X/E$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in A \text{ and } x \neq b, \\ a, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_{\mathrm{E}}(X)$, let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in B$ for some $B \in X/E$.

Case 1. $B \neq A$. This implies that $x, y \notin A$. Hence $(x\alpha, y\alpha) = (a, a) \in E$.

Case 2. $B = A$. If $b \in A$, then $x\alpha$, $y\alpha \in \{a, b\} \subseteq A$. Thus $(x\alpha, y\alpha) \in E$. If $b \notin A$, then $x \neq b$ and $y \neq b$ which implies that $(x\alpha, y\alpha) = (b, b) \in E$.

From two cases, we then have $\alpha \in T_{\mathrm{E}}(X)$. Since $a < b$ and $a\alpha = b \nleqslant a = b\alpha$, we get $\alpha \notin T_{\mathrm{O}}(X)$. Hence $T_{\mathrm{EO}}(X) \neq T_{\mathrm{E}}(X)$. □

Theorem 7 $T_{\mathrm{SEO}}(X) = T_{\mathrm{SE}}(X)$ *if and only if* $E = I_X$.

Proof: Assume that $E \neq I_X$. Then there exist $a, b \in X$ such that $(a, b) \in E$ and $a \neq b$. We may assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = a, \\ a, & \text{if } x = b, \\ x, & \text{otherwise.} \end{cases}$$

It is easy to verity that $\alpha \in T_{\mathrm{SE}}(X)$. Since $a < b$ and $a\alpha > b\alpha$, $\alpha \notin T_{\mathrm{O}}(X)$. Therefore $T_{\mathrm{SEO}}(X) \neq T_{\mathrm{SE}}(X)$.

Conversely, suppose that $E = I_X$. Then we get $T_{\mathrm{SE}}(X) = \{i_X\}$. It follows that $T_{\mathrm{SEO}}(X) = T_{\mathrm{SE}}(X) \cap T_{\mathrm{O}}(X) = \{i_X\} \cap T_{\mathrm{O}}(X) = \{i_X\} = T_{\mathrm{SE}}(X)$. □

Theorem 8 $T_{\mathrm{SEO}}(X) = T_{\mathrm{O}}(X)$ *if and only if* $E = X \times X$.

Proof: Suppose that $E \neq X \times X$. Then there exist $a, b \in X$ such that $(a, b) \notin E$. We may assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \geqslant a, \\ a, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_{\mathrm{O}}(X)$, let $x, y \in X$ be such that $x \leqslant y$.

Case 1. If $a \leqslant x \leqslant y$ or $x \leqslant y < a$, then we get $x\alpha = y\alpha$.

Case 2. If $x < a \leqslant y$, then we have that $x\alpha = a < b = y\alpha$.

From two cases, we deduce that $\alpha \in T_{\mathrm{O}}(X)$. Since $(a, a\alpha) = (a, b) \notin E, \alpha \notin T_{\mathrm{SE}}(X)$. Hence $T_{\mathrm{SEO}}(X) \neq T_{\mathrm{O}}(X)$.

Conversely, assume that $E = X \times X$. Thus $T_{\mathrm{SE}}(X) = T(X)$. Hence $T_{\mathrm{SEO}}(X) = T_{\mathrm{SE}}(X) \cap T_{\mathrm{O}}(X) = T(X) \cap T_{\mathrm{O}}(X) = T_{\mathrm{O}}(X)$. □

Theorem 9 $T_{\mathrm{OR}}(X) = T_{\mathrm{O}}(X)$ *if and only if* $|X| = 1$.

Proof: Assume that $|X| > 1$. Then there exist $a, b \in X$ such that $a < b$. Define $\alpha : X \to X$ as given in Theorem 8. Then $\alpha \in T_{\mathrm{O}}(X)$. Since $a\alpha = b > a, \alpha \notin T_{\mathrm{R}}(X)$. Hence $T_{\mathrm{OR}}(X) \neq T_{\mathrm{O}}(X)$. □

Theorem 10 $T_{\mathrm{OR}}(X) = T_{\mathrm{R}}(X)$ *if and only if* $|X| \leqslant 2$.

Proof: Suppose that $|X| > 2$. Let $a, b, c \in X$ be such that $a < b < c$. We define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = c, \\ x, & \text{otherwise.} \end{cases}$$

Then $\alpha \in T_{\mathrm{R}}(X)$. We note that $b < c$ and $b\alpha = b \nleqslant a = c\alpha$. Thus $\alpha \notin T_{\mathrm{O}}(X)$ and so $T_{\mathrm{OR}}(X) \neq T_{\mathrm{R}}(X)$.

Conversely, assume that $|X| \leqslant 2$. Let $\alpha \in T_{\mathrm{R}}(X)$. To show that $\alpha \in T_{\mathrm{OR}}(X)$, let $x, y \in X$ be such that $x < y$. Since $\alpha \in T_{\mathrm{R}}(X)$ and by assumption, we then have $x\alpha = x$ (since $x\alpha \leqslant x$). It follows that $x\alpha \leqslant y\alpha$ (since $y\alpha = x$ or $y\alpha = y$), which implies that $\alpha \in T_{\mathrm{O}}(X)$. Therefore $T_{\mathrm{OR}}(X) = T_{\mathrm{R}}(X)$. □

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