Regularity criterion for weak solutions to the 3D Boussinesq equations

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ABSTRACT: In this paper, a regularity criterion for the 3D Boussinesq equations is investigated. We prove that for some $T > 0$ if $\int_0^T \|u\|_{L^3}^\beta \, dt < \infty$, where $3/\alpha + 2/\beta \leq 1$ and $\alpha \geq 3$, then the solution $(u, \theta)$ can be extended smoothly beyond $t = T$. The derivative $u_x$ can be replaced by any directional derivative of $u$.

KEYWORDS: global regularity, energy estimate

INTRODUCTION

In the paper we investigate the initial value problem for the Boussinesq equations in $\mathbb{R}^3$

$$
\begin{align}
\partial_t u - \mu \Delta u + u \cdot \nabla u + \nabla \Pi &= \theta \hat{z}, \quad (1a) \\
\partial_t \theta - \nu \Delta \theta + u \cdot \nabla \theta &= 0, \quad (1b) \\
\nabla \cdot u &= 0 \quad (1c)
\end{align}
$$

with the initial value

$$
t = 0 : \quad u = u_0(x), \quad \theta = \theta_0(x). \quad (2)
$$

where $u$ is the velocity, $\Pi$ is the pressure, $\theta$ is the temperature, $\mu$ is the viscosity, $\nu$ is called the molecular diffusivity, and $\hat{z} = (0, 0, 1)^T$.

The Boussinesq equations are of relevance in studying a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role\textsuperscript{1}. The scalar function $\theta$ may for instance represent temperature variation in a gravity field, and $\theta \hat{z}$ the buoyancy force. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research and many interesting results have been obtained; see Refs. 2–4 for regularity criteria and see Refs. 5–7 for blow up criteria.

If $\theta = 0$, then (1) reduce to the Navier-Stokes equations. Besides their physical applications, the Navier-Stokes equations are also mathematically significant. Leray\textsuperscript{8} and Hopf\textsuperscript{9} constructed weak solutions to the Navier-Stokes equations. The solution is called the Leray-Hopf weak solution. From then on, much effort has been devoted to establishing the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Lots of regularity criteria of weak solutions have been proposed and many interesting results have been established (e.g., regularity criteria via the velocity\textsuperscript{10–12}, via the derivative of the velocity in one or two direction\textsuperscript{13–15}, and via the pressure\textsuperscript{16–18}). Logarithmically improved regularity criteria of weak solutions were established in Refs. 19, 20.

The purpose of this paper is to establish the regularity criteria of weak solutions to (1) and (2) via the derivative of the velocity in one direction. We first give the definition of a weak solution.

Definition 1. Let $u_0, \theta_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Then $(u, \theta)$ is called a weak solution of (1) and (2) if the following conditions are satisfied.

(i) $u, \theta \in L^2(0, T; H^1(\mathbb{R}^3)) \cap C_w(0, T; L^2(\mathbb{R}^3))$ with $\nabla \cdot u = 0$ in the sense of distribution.

(ii) For any $\varphi, \psi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$ and $\varphi(T, \cdot) = 0$, $\psi(T, \cdot) = 0$, there holds

$$
\int_0^T \int_{\mathbb{R}^3} (-u \varphi_t + \mu \nabla u \cdot \nabla \varphi - u(u \cdot \nabla) \varphi) \, dr \, dt
= \int_0^T \int_{\mathbb{R}^3} \theta \hat{z} \cdot \varphi \, dr \, dt + \int_{\mathbb{R}^3} u_0 \varphi(0, \cdot) \, dr,
$$

$$
\int_0^T \int_{\mathbb{R}^3} (-\theta \psi_t + \nu \nabla \theta \cdot \nabla \psi - \theta u \cdot \nabla \psi) \, dr \, dt
= \int_{\mathbb{R}^3} \theta_0 \psi(0, \cdot) \, dr.
$$


Proof of Main Result

For given initial data $u_0, \theta_0 \in H^1(\mathbb{R}^3)$, the weak solution is the same as the local strong solution $(u, \theta)$ in a local interval $(0, T)$ as in the discussion of the Navier-Stokes equations. Thus Theorem 1 is reduced to establishing a priori estimates uniformly in $(0, T)$ for strong solutions. With the use of the a priori estimates, the local strong solution $(u, \theta)$ can be continuously extended to $t = T$ by a standard process to obtain global regularity of the weak solution. Therefore, we assume that the solution $(u, \theta)$ is sufficiently smooth on $(0, T)$.

Multiplying (1b) by $\theta$ and integrating with respect to $r = (x, y, z)$ on $\mathbb{R}^3$, and using integration by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 = 0.$$  \hspace{1cm} (7)

Similarly, we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \int_{\mathbb{R}^3} (\theta \hat{z}) u \, dr.$$  \hspace{1cm} (8)

Summing (7) and (8) and using Cauchy’s inequality, we deduce that

$$\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \right) + \|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 \leq \frac{1}{2} \left( \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \right).$$  \hspace{1cm} (9)

It follows from (9) and Gronwall’s inequality that

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2 \int_0^t \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \, d\tau \leq C \left( \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \right).$$  \hspace{1cm} (10)

Differentiating (1a) and (1b) with respect to $z$, we obtain

$$\partial_t u_z - \Delta u_z + u_z \cdot \nabla u + u \cdot \nabla u_z + \nabla \Pi_z = \theta \hat{z},$$  \hspace{1cm} (11a)

$$\partial_t \theta_z - \Delta \theta_z + u_z \cdot \nabla \theta + u \cdot \nabla \theta_z = 0.$$  \hspace{1cm} (11b)

Taking the inner product of $u_z$ with (11a) and using integration by parts yields

$$\frac{1}{2} \frac{d}{dt} \|u_z(t)\|_{L^2}^2 + \|\nabla u_z(t)\|_{L^2}^2 = - \int_{\mathbb{R}^3} u_z \cdot \nabla u \, dz + \int_{\mathbb{R}^3} (\theta \hat{z}) \cdot u_z \, dz.$$  \hspace{1cm} (12)

Similarly, we get

$$\frac{1}{2} \frac{d}{dt} \|\theta_z(t)\|_{L^2}^2 + \|\nabla \theta_z(t)\|_{L^2}^2 = - \int_{\mathbb{R}^3} u_z \cdot \nabla \theta \cdot \theta_z \, dz.$$  \hspace{1cm} (13)
Combining (12) and (13) yields
\[
\frac{1}{2} \frac{d}{dt} \left( \| u_z(t) \|_{L^2}^2 + \| \theta_z(t) \|_{L^2}^2 \right)
+ \| \nabla u_z(t) \|_{L^2}^2 + \| \nabla \theta_z(t) \|_{L^2}^2
= - \int_{\mathbb{R}^3} u_z \cdot \nabla u \cdot u_z \, dr + \int_{\mathbb{R}^3} (\theta z)_z \cdot u_z \, dr
- \int_{\mathbb{R}^3} u_z \cdot \nabla \theta \cdot \theta_z \, dr
\]
\[\triangleq I_1 + I_2 + I_3. \tag{14}\]

In what follows, we estimate the \( I_j \). By integration by parts and Hölder’s inequality, we obtain
\[
I_1 \leq C \| \nabla u_z \|_{L^2} \| u_z \|_{L^\alpha} \| u \|_{L^\infty}^{\alpha-1},
\]
where
\[
\frac{1}{\sigma} + \frac{1}{3\alpha} = \frac{1}{2}, \quad 2 \leq \sigma \leq 6.
\]
It follows from the interpolating inequality that
\[
\| u_z \|_{L^\sigma} \leq C \| u_z \|_{L^2}^{1-3\left(\frac{1}{2} - \frac{1}{\sigma}\right)} \| \nabla u_z \|_{L^\infty}^{3\left(\frac{1}{2} - \frac{1}{\sigma}\right)}.
\]

From (5), we get
\[
I_1 \leq C \| \nabla u_z \|_{L^2} \| u_z \|_{L^2}^{1-3\left(\frac{1}{2} - \frac{1}{\sigma}\right)} \times \| \nabla u_z \|_{L^2}^{3\left(\frac{1}{2} - \frac{1}{\sigma}\right)} \| u_z \|_{L^\infty} \frac{1}{2}
\]
\[\leq C \| \nabla u_z \|_{L^2}^{1+3\left(\frac{1}{2} - \frac{1}{\sigma}\right)} \| u_z \|_{L^2} \times \| \nabla u_z \|_{L^\infty}^{\frac{1}{2}} \| u_z \|_{L^2},
\]
\[\leq \frac{1}{2} \| \nabla u_z \|_{L^2}^2 + C \| u_z \|_{L^2}^2 \| \nabla u_z \|_{L^\infty}^{2 q} \| u_z \|_{L^\infty}^q,
\]
where
\[
q = \frac{2}{3 - 3\left(\frac{1}{2} - \frac{1}{\sigma}\right)} = \frac{2}{3\left(1 - \frac{1}{\alpha}\right)}.
\]
When \( \alpha \geq 3 \), we have \( 2q \leq 2 \) and application of Young’s inequality yields
\[
I_1 \leq \frac{1}{2} \| \nabla u_z \|_{L^2}^2 + C \| u_z \|_{L^2}^2 (\| \nabla u_z \|_{L^2}^2 + \| u_z \|_{L^\infty}^q), \tag{15}\]
where
\[
\frac{3}{\alpha} + \frac{2}{q} = 1.
\]
By Cauchy’s inequality, we have
\[
I_2 \leq \frac{1}{2} (\| u_z \|_{L^2}^2 + \| \theta_z \|_{L^2}^2), \tag{16}\]
From Hölder’s inequality, we obtain
\[
I_3 \leq C \| \nabla \theta_z \|_{L^2} \| \theta_z \|_{L^\frac{2\alpha-3}{\alpha}} \| u_z \|_{L^\infty}
\leq C \| \nabla \theta_z \|_{L^2} \| u_z \|_{L^\infty} (\| \theta_z \|_{L^\frac{2\alpha-3}{\alpha}}) \| \nabla \theta_z \|_{L^\frac{2\alpha-3}{\alpha}}
\leq \frac{1}{2} \| \nabla \theta_z \|_{L^2}^2 + C (\| \nabla \theta_z \|_{L^2}^2 + \| u_z \|_{L^\infty}^q) \| \theta_z \|_{L^\frac{2\alpha-3}{\alpha}}. \tag{17}\]
Combining (14)–(17) yields
\[
\frac{d}{dt} (\| u_z \|_{L^2}^2 + \| \theta_z \|_{L^2}^2) + \| \nabla u_z \|_{L^2}^2 + \| \nabla \theta_z \|_{L^2}^2
\leq C (\| u \|_{L^2}^2 + \| \theta \|_{L^2}^2) (\| \nabla u_z \|_{L^2}^2 + \| u_z \|_{L^\infty}^q + 1)
+ C (\| \nabla \theta \|_{L^2}^2 + \| u_z \|_{L^\infty}^q) \| \theta_z \|_{L^\frac{2\alpha-3}{\alpha}}.
\]
From Gronwall’s inequality, we get
\[
\| u_z \|_{L^2}^2 + \| \theta_z \|_{L^2}^2 + s \int_0^t \| \nabla u_z \|_{L^2}^2 \, d\tau
\]
\[+ \int_0^t \| \nabla \theta_z \|_{L^2}^2 \, d\tau
\leq C e (\| u_0 \|_{L^2}^2 + \| \theta_0 \|_{L^2}^2) e (\| \theta_0 \|_{H^1} + \| \theta \|_{H^1}^2)
\]
\[C (\| u_0 \|_{L^2}^2 + \| \theta_0 \|_{L^2}^2 + \Theta (t) \| \theta \|_{H^1}^{\frac{2\alpha-3}{\alpha}}). \tag{18}\]

Multiplying (1a) by \(-\Delta u\) and integrating with respect to \( x \) on \( \mathbb{R}^3 \), then using integration by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|_{L^2}^2 + \| \Delta u \|_{L^2}^2
= \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta u \, dr - \int_{\mathbb{R}^3} (\theta z) \cdot \Delta u \, dr. \tag{19}\]
Similarly, we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta(t) \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2
= \int_{\mathbb{R}^3} u \cdot \nabla \theta \cdot \Delta \theta \, dr. \tag{20}\]
Collecting (19) and (20) yields
\[
\frac{1}{2} \frac{d}{dt} (\| \nabla u(t) \|_{L^2}^2 + \| \nabla \theta(t) \|_{L^2}^2) + \| \Delta u \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2
= \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta u \, dr - \int_{\mathbb{R}^3} (\theta z) \cdot \Delta u \, dr
\]
\[+ \int_{\mathbb{R}^3} u \cdot \nabla \theta \cdot \Delta \theta \, dr
\triangleq J_1 + J_2 + J_3. \tag{21}\]
In what follows, we estimate the \( J_i \). By (6) and
Young’s inequality, we deduce that

\[
J_1 \leq C \| \nabla u \|_{L^3}^3 \\
\leq C \| \nabla u \|_{L^2}^2 \| \nabla \nabla u \|_{L^2} \| \nabla u_z \|_{L^2}^{\frac{1}{2}} \\
\leq \frac{1}{4} \| \nabla \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^3 \| \nabla u_z \|_{L^2} \\
\leq \frac{1}{4} \| \nabla \nabla u \|_{L^2}^2 + C (\| \nabla u \|_{L^2} + \| \nabla u_z \|_{L^2}) \\
x \| \nabla u \|_{L^2}^2, \tag{22}
\]

where \( \nabla \perp = (\partial_x, \partial_y) \).

Using integration by parts and Cauchy’s inequality, we have

\[
J_2 \leq \frac{1}{2} (\| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2). \tag{23}
\]

By (6) and Young’s inequality, we have

\[
J_3 \leq \| \nabla u \|_{L^3} \| \nabla \theta \|_{L^3}^2 \\
\leq C \| \nabla u \|_{L^2}^\frac{1}{2} \| \nabla \nabla u \|_{L^2}^\frac{1}{2} \| \nabla u_z \|_{L^2}^\frac{1}{2} \\
\times \| \nabla \theta \|_{L^2}^2 \| \nabla \nabla \theta \|_{L^2}^2 \| \nabla \theta_z \|_{L^2}^2 \\
\leq \frac{1}{4} \| \nabla \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla u_z \|_{L^2}^2 \\
\times \| \nabla \theta \|_{L^2}^2 \| \nabla \nabla \theta \|_{L^2}^2 \| \nabla \theta_z \|_{L^2}^2 \\
\leq \frac{1}{4} \| \nabla \nabla u \|_{L^2}^2 + \frac{1}{2} \| \nabla \nabla \theta \|_{L^2}^2 \\
\times \| \nabla \theta \|_{L^2}^2 \| \nabla \nabla \theta \|_{L^2}^2 \| \nabla \theta_z \|_{L^2}^2 \\
\leq \frac{1}{4} \| \nabla \nabla u \|_{L^2}^2 + \frac{1}{2} \| \nabla \nabla \theta \|_{L^2}^2 \\
\times (\| \nabla u \|_{L^2}^2 + \| \nabla u_z \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 + \| \nabla \theta_z \|_{L^2}^2). \tag{24}
\]

Combining (21)–(24) yields

\[
\frac{d}{dt} (\| \nabla u(t) \|_{L^2}^2 + \| \nabla \theta(t) \|_{L^2}^2) + \| \Delta u \|_{L^2}^2 + \| \Delta \theta \|_{L^2}^2 \\
\leq C (\| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2)(\| \nabla u \|_{L^2}^2 \\
+ \| \nabla u_z \|_{L^2}^2 + \| \nabla \theta_z \|_{L^2}^2 + 1). \tag{25}
\]

From (25), Gronwall’s inequality, (10), and (18), we know that \((u, \theta) \in L^\infty(0, T; H^1(\mathbb{R}^3))\). Thus \((u, \theta)\) can be extended smoothly beyond \(t = T\). We have completed the proof of Theorem 1.

Acknowledgements: The authors would like to thank the referee for their pertinent comments and advice. This work was supported in part by the NNSF of China (Grant No. 11101144) and Research Initiation Project for High-level Talents (201031) of North China University of Water Resources and Electric Power.

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