RESEARCH ARTICLE

ScienceAsia 36 (2010): **244–248**

doi: 10.2306/scienceasia1513-1874.2010.36.244

Double bubbles outside a disc

Banyat Sroysang[a], **Wacharin Wichiramala**[b,c,*]

[a] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Rangsit Centre, Pathumthani 12121, Thailand

[b] Department of Mathematics, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

[c] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand

*Corresponding author, e-mail: wacharin.w@chula.ac.th

Received 30 Apr 2010
Accepted 14 Aug 2010

ABSTRACT: We show that the least-perimeter way to enclose and separate two regions of prescribed area outside a disc is a truncated standard double bubble.

KEYWORDS: soap bubble, minimizing enclosure, least-perimeter enclosing

INTRODUCTION

The soap bubble problem is a generalization of the classical isoperimetric problem. Specifically, the planar soap bubble problem is the search for the least-perimeter way to enclose and separate m regions on the plane of a given m areas. More precisely, for given $A_1, \cdots, A_m > 0$, we expect to find minimizing enclosures of regions $R_1, \cdots, R_m$ of areas $A_1, \cdots, A_m$, respectively. Existence and regularity of minimizing enclosures on the plane is provided by Ref. 1. Enclosures with this nice regularity are called *bubbles*. In higher dimensions, the existence is shown in Ref. 3. It is natural to believe that all regions of a minimizer must be connected. But this turns out to be the most difficult part of this problem. Therefore the main conjecture of the bubble problem is that a minimizing bubble has connected regions. For simplicity, we say a bubble is *standard* if its regions including the exterior region are connected.

On the plane, the case of 4 areas ($m = 4$) is still open. For a single area, a circle is the unique shortest enclosure. For two areas, Foisy et al concluded that the standard double bubble (see Fig. 1) is uniquely minimizing[4]. For three areas, Wichiramala settled the conjecture that the standard triple bubble (Fig. 1) is the shortest uniquely[5,6].

In higher dimensions, in $\mathbb{R}^n$ for $n \geqslant 3$, we just have results for double bubbles where we look for the least $(n-1)$-dimensional measure way to enclose and separate 2 regions of 2 given n-dimensional volumes. Hutchings et al[7] proved the double bubble conjecture in $\mathbb{R}^3$. Reichardt et al[8] showed this for $n = 4$, and finally, Reichardt[9] showed this for $n > 4$.

Many variants of the problem are studied in many domains. Studies have been done for cases of the double bubble on a half plane[10], on a cone[10], on a flat 2-torus[11], on a flat 3-torus[12], in a disc[13], in Gauss space and spheres[14], in the spherical space and the hyperbolic space[15–17].

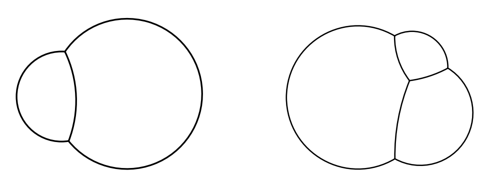

Fig. 1 A standard double bubble (*left*) and triple bubble (*right*).

In this work, we study minimizing enclosures of 2 given areas on the complement D^C of a closed disc D. An enclosure E separates D^C into regions. Each region is a union of connected and open components of $D^C \setminus E$. We say E enclosures regions of areas $A_1, \cdots, A_m$ if $D^C \setminus E$ has regions $R_1, \cdots, R_m$ of areas $A_1, \cdots, A_m$. The exterior region $D^C \setminus \cup_i \overline{R_i}$ is denoted by R_0. It is clear that part of the boundary ∂D of D is not needed as a part of an enclosure.

We expect to prove that the truncated standard double bubble (Fig. 2) is the shortest enclosure for 2 given areas. In the process, we use the weak approach from Refs. 4–6, 18. For the three-dimensional case, we intuitively believe that the bubbles in Fig. 3 are optimal.

BASIC RESULTS

In this section, we list all basic results for minimizing enclosures on D^C. We start with the existence and regularity of the minimizers and then their basic prop-

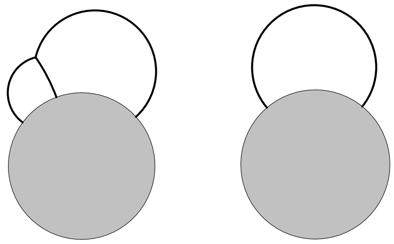

Fig. 2 A truncated standard double bubble (*left*) and a truncated circle (*right*) outside a disc.

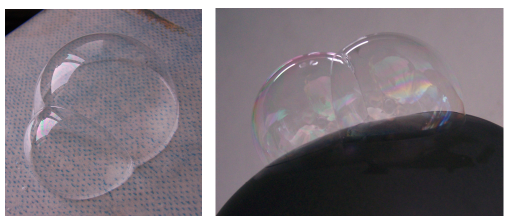

Fig. 3 Efficient double bubbles on a plane (*left*) and on a sphere (*right*).

erties.

For minimizing enclosures on the plane, the existence and regularity are completed in Ref. 1. From the argument in Refs. 1, 2, 13, we obtain the following existence and regularity theorem.

Theorem 1 *For given $A_1, \cdots, A_m > 0$, there exists a minimizing enclosure on D^C of areas $A_1, \cdots, A_m$. Each minimizer is composed of finitely many circular or straight edges separating pairs of different regions. These edges meet in threes at 120° angles or one edge meets ∂D perpendicularly. There are real numbers $p_1, \cdots, p_m$, called pressures, such that each edge between R_i and R_j has curvature $|p_i - p_j|$ and curves into the lower pressure region where p_0 is set to be 0.*

In the following figures, the numbers 1,2,...,m,0 in each component will indicate the region this component contributes area to. We also relabel regions so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$.

Lemma 1 *A minimizing enclosure on D^C is attached to D and is a connected graph.*

Proof: Let B be a minimizing enclosure. If B is not attached to D or not a connected graph, we can translate or slide some part of B around D to create an illegal meeting of edges and hence create a nonminimizing enclosure with the same length and areas. This contradicts the minimality of B. □

We now can conclude easily that the following theorem holds.

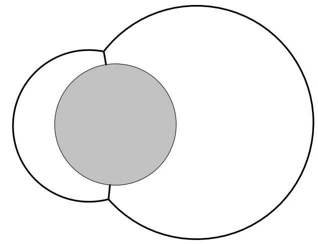

Fig. 4 A double bubble with connected regions completely surrounding the disc.

Theorem 2 *A minimizing enclosure of a single area on D^C is a truncated circle as illustrated in Fig. 2.*

From Ref. 19, we may conclude directly that the following lemma holds.

Lemma 2 *For a minimizing enclosure on D^C, any 2 components may meet at most once.*

We then can conclude, as a corollary, the next lemma which similar to the one from Ref. 4.

Lemma 3 *For a minimizing enclosure of many areas on D^C, there is no 2-sided component.*

By the same argument as in Ref. 19 we obtain the following lemma.

Lemma 4 *The enclosure in Fig. 4 is not minimizing.*

Proof: As the 2 internal edges are on a circle, D can be moved along the circle to create an illegal meeting. □

We define a *bubble* to be an enclosure with properties in Theorem 1, Lemma 1 and Lemma 2. Hence a bubble of many areas has no 2-sided component.

By the argument from Ref. 2, we have the following result.

Lemma 5 *Let B_t be a variation of a bubble B on D^C. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} l(B_t)|_{t=0} = \sum_i p_i \frac{\mathrm{d}}{\mathrm{d}t} A_i^t|_{t=0}.$$

In Lemma 5.9 of Ref. 6 (Lemma 5.11 of Ref. 5), we show that a 3-sided component of a bubble on the plane can be 'reduced' to get another bubble. By a similar argument, the following lemma shows that we may reduce a 3-sided component attached to D (see Fig. 5) to get another bubble.

Lemma 6 *Let C be a 3-sided component of a bubble attached to D. Then we can prolong the incident edge of C into C so that it is perpendicular to ∂D.*

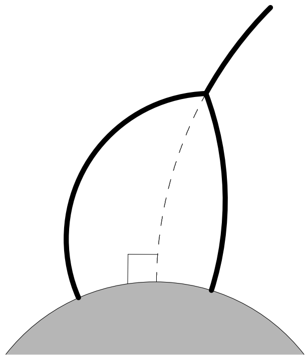

Fig. 5 The incident edge of a 3-sided component can be prolonged inside.

Lemma 7 *A truncated standard double bubble on D^C has positive pressures.*

Proof: From the previous lemma, the external edge of R_2 can be prolonged into R_1 and meet ∂D perpendicularly. Hence the edge is on a circle bounding R_2 inside. Therefore $p_1 \geqslant p_2 > 0$ as desired. □

WEAK APPROACH

The weak approach helps making R_0 connected by simply allowing a bubble to enclose areas greater than $A_1, \cdots, A_m$. The idea was originated in Ref. 4 and then completed in Refs. 5, 6. We define a weak enclosure for areas $A_1, \cdots, A_m$ to be an enclosure of area $a_1, \cdots, a_m$ where $a_i \geqslant A_i$. We will show existence of minimizing weak enclosures and list their properties later.

Let $L(A_1, \cdots, A_m)$ be the length of a minimizing bubble on D^C of areas $A_1, \ldots, A_m$. Since $L(A_1, \cdots, A_m)$ is continuous and tends to infinity as each A_i approaches infinity, we have that $\min_{a_i \geqslant A_i} L(A_1, \cdots, A_m)$ exists. Equivalently, a minimizing weak enclosure exists and is a minimizing enclosure of areas it encloses. Hence these minimizers are also bubbles. Consequently, we may call them *minimizing weak bubbles* or *weakly minimizing bubbles*.

By the argument in Proposition 3.5 of Ref. 6 (Proposition 3.6 of Ref. 5), we may obtain the following lemma.

Lemma 8 *A weak minimizer on D^C has connected R_0 and non-negative pressures. Moreover, if $p_i > 0$, then R_i has area A_i.*

The advantage of using the weak approach is that mainly we just have to show that a weak minimizer has the property we desire. Then we can conclude easily

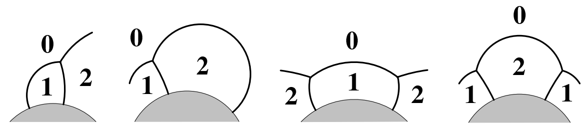

Fig. 6 All possible components of weak minimizers.

that a minimizer has that desired property too. As weakly minimizing is a stronger condition than being minimizing, we have to get rid of a smaller class of candidative bubbles. Moreover, in order to get rid of unwanted bubbles, we may find a shorter enclosure that is allowed to enclose greater areas.

SHAPES OF WEAKLY MINIMIZING DOUBLE BUBBLES ON D^C

In this section, we will list crucial properties of weak minimizers that will be used in the next section. Note again that R_0 is connected and all pressures are nonnegative. For convenience, we suppose that D is centred at the origin.

From Lemma 2 and the fact that R_0 is connected, a component of R_1 and R_2 may meet R_0 at most once and it may not be 2-sided. Hence it must be attached to D and has at most 4 sides as listed in Fig. 6. We say a 4-sided component of R_1 is *circular* if it has 2 (opposite) edges on a circle. It is clear that every 4-sided component of R_2 is symmetric.

Lemma 9 *Let B be a weakly minimizing double bubble on D^C. Suppose that D meets R_0. Then (1) every external edge is on a circle meeting ∂D perpendicularly, (2) every 4-sided component is symmetric, (3) all 4-sided components of R_2 are isometric, and (4) for R_2, a 4-sided component can be fit tightly in a 3-sided component, if it exists. In other words, we can trim a 3-sided component to get a 4-sided component.*

Proof: Since D meets R_0, B is composed of consecutive components such that the first and the last components are 3-sided with possible some 4-sided components in between. By Lemma 6, we have (1). Then (2) and (3) are clear. Now consider components of R_2. Due to its symmetry, a 4-sided component may be obtained by cutting off a 3-sided component. Hence we have (4). □

Lemma 10 *For a weakly minimizing double bubble on D^C, every external edge of R_2 is not centred at the origin.*

Proof: According to Fig. 7, a circular arc centred at the origin would make an acute angle with the edge between R_1 and R_2. Hence we are done. □

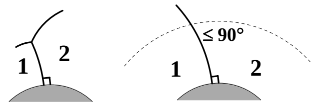

Fig. 7 An external edge of R_2.

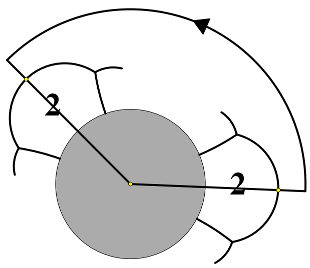

Fig. 8 A rotating variation of the bold part.

Lemma 11 *A weakly minimizing double bubble on D^C may not have 2 isometric 4-sided components of R_2.*

Proof: Suppose there exist 2 such isometric components. Consider the rotating variation on the bold part around the origin in Fig. 8. This variation preserves areas of R_1 and R_2. By the variation argument from Ref. 13 and using the argument from Lemma 4.11 in Ref. 6 (Lemma 4.19 in Ref. 5), the 2 external edges of R_2 are centred at the origin, a contradiction to Lemma 9. □

Proposition 1 *Let B be a weak minimizing double bubble on D^C. If D meets R_0, then B is standard.*

Proof: Suppose D meets R_0 and B is not standard. From Lemma 9, components of R_2 have the following properties: (1) 4-sided components are isometric, and (2) a 4-sided component is a part of a 3-sided component. By Lemma 11, there is at most one component of R_2. If R_2 is 3-sided, then B is standard. Hence R_2 is 4-sided and B is illustrated by Fig. 9. Consider the rotated variation on the bold part around the centre of the external edge of R_2 in Fig. 9. This variation preserves areas of R_1 and R_2. By the same variation argument as used in the proof of Lemma 11, the 4 edges of R_1 are concentric, a contradiction. Therefore B must be standard. □

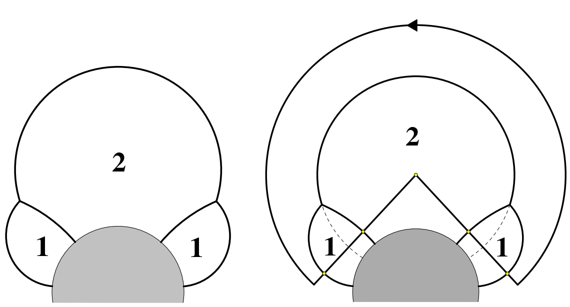

Fig. 9 A bubble with 3 components (*left*); a rotated variation on this bubble (*right*).

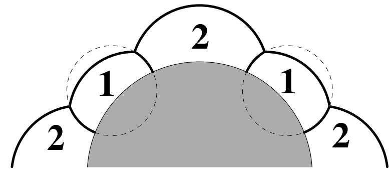

Fig. 10 A 4-sided component of R_2 between 2 circular components of R_1.

Proposition 2 *A weak minimizing double bubble on D^C is standard.*

Proof: Suppose to get a contradiction that B is not standard. From the previous lemma, D is completely surrounded by $R_1 \cup R_2$. Hence every component of R_1 and R_2 is 4-sided and R_1 and R_2 have the same numbers of components surrounding D alternately. First suppose that there is a symmetric 4-sided component C of R_1. From Lemma 4, R_2 has at least 2 components. Consider the 2 components D_1 and D_2 of R_2 next to C. Note that every 4-sided component of R_2 is symmetric. By Lemma 10, their external edges are not centred at the origin. By Lemma 5.3 of Ref. 6 (Lemma 5.5 of Ref. 5), D_1 and D_2 are isometric. By Lemma 11, B is not minimizing. Hence every component of R_1 is circular. We will divide into cases according to the number of components of R_2.

Case R_2 has at least 3 components. In Fig. 10 the 2 circular components of R_1 are isometric. Consider the left component D_1 and the right component D_2 of R_2. Again, as they are symmetric and their external edges are not centred at the origin, they are isometric. Thus B is not minimizing by Lemma 11.

Case R_2 has 2 components. Here B is illustrated by Fig. 11. By Lemma 5.19 of Ref. 6 (Lemma 5.35 of Ref. 5), the 2 external edges of R_2 are on a circle. By the same argument as in Lemma 5.22 of Ref. 6 (Lemma 5.38 of Ref. 5), we can create a bubble of the same length and area by moving the 2 components of R_1 towards each other and eventually create an illegal meeting. Therefore B is not minimizing.

In both cases, we encounter contradictions. Thus B must be standard. □

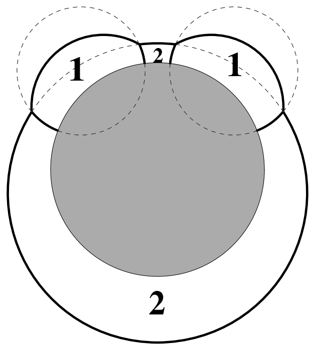

Fig. 11 R_1 has 2 components and they are circular.

MAIN RESULT

Now we conclude with the main theorem.

Theorem 3 *Every minimizing double bubble on* D^C *is standard.*

Proof: Let B be a minimizing double bubble of areas A_1 and A_2. Let W be a weakly minimizing double bubble for areas A_1 and A_2. From the previous theorem, W is standard. By Lemma 7, both regions of W have positive pressures. By Lemma 8, the regions of W have areas A_1 and A_2. Hence W is a minimizer. Since $l(B) = l(W)$, B is also a weak minimizer. Therefore B is standard by the previous theorem. □

We may apply our method to get the same result for a flat cover of the complement of the disc as follows.

Theorem 4 *In any flat cover of the complement of the disc, the minimizing enclosure of 2 given areas remains the truncated standard double bubble.*

Proof: For a bubble in this domain, the boundary always meets R_0. Hence we may conclude easily that the truncated standard double bubble is minimizing. □

Acknowledgements: We would like to thank Joel Foisy for comments, and the referees for comments and suggestions. This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

REFERENCES

1. Morgan F (1994) Soap bubbles in $\mathbb{R}^2$ and on surfaces. *Pac J Math* **165**, 347–61.
2. Cox C, Harrison L, Hutchings M, Kim S, Light J, Mauer A, Tilton M (1994/95) The shortest enclosure of three connected areas in $\mathbb{R}^2$. *Real Anal Exchange* **20**, 313–35.
3. Almgren FJ (1976) Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem Am Math Soc* **165**.
4. Foisy J, Alfaro M, Brock J, Hodges N, Zimba J (1993) The standard double soap bubble in $\mathbb{R}^2$ uniquely minimizes perimeter. *Pac J Math* **159**, 47–59.
5. Wichiramala W (2002) The planar triple bubble problem. PhD thesis, Univ of Illinois at Urbana-Champaign.
6. Wichiramala W (2004) Proof of the planar triple bubble conjecture. *J Reine Angew Math* **567**, 1–49.
7. Hutchings M, Morgan F, Ritoré F, Ros A (2002) Proof of the double bubble conjecture. *Ann Math* **155**, 459–89.
8. Reichardt BW, Heilmann C, Lai Y, Spielman A (2003) Proof of the double bubble conjecture in $\mathbb{R}^4$ and certain higher dimensional cases. *Pac J Math* **208**, 347–66.
9. Reichardt BW (2008) Proof of the double bubble conjecture in $\mathbb{R}^n$. *J Geom Anal* **18**, 172–91.
10. Lopez R, Baker TB (2006) The double bubble problem on cone. *New York J Math* **12**, 157–67.
11. Corneli J, Holt P, Lee G, Leger N, Schoenfeld E, Steinhurst B (2004) The double bubble problem on the flat two-torus. *Trans Am Math Soc* **356**, 3769–820.
12. Álvarez MC, Corneli J, Walsh G, Beheshti S (2003) Double bubbles in the three-torus. *Exp Math* **12**, 79–89.
13. Cañete A, Ritoré M (2004) Least-perimeter partitions of the disk into three regions of given areas. *Indiana Univ Math J* **53**, 883–904.
14. Corneli J, Corwin I, Xu Y, Hurder S, Sesum V, Adams E, Davis D, Lee M, Pettit R, Hoffman N (2008) Double bubbles in Gauss space and spheres. *Houston J Math* **34**, 181–204.
15. Masters JD (1996/97) The perimeter-minimizing enclosure of two areas in S^2. *Real Anal Exchange* **22**, 645–54.
16. Cotton A, Freeman D (2002) The double bubble problem in spherical and hyperbolic space. *Int J Math Sci* **32**, 641–99.
17. Corneli J, Hoffman N, Holt P, Lee G, Leger N, Moseley S, Schoenfeld E (2007) Double bubbles in S^3 and H^3. *J Geom Anal* **17**, 189–212.
18. Vaughn R (1998) Planar soap bubbles. PhD thesis, Univ of California, Davis.
19. Bleicher MN (1996) Isoperimetric networks in the Euclidean plane. *Stud Sci Math Hung* **31**, 455–78.