SHORT REPORT

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A note on the existence of the integers and rationals

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ABSTRACT: We examine the role of the replacement axiom and the power set axiom in proving the existence of the integers ($\mathbb{Z}$) and the rationals ($\mathbb{Q}$). We show that without the power set axiom, the replacement axiom is sufficient for the existence of $\mathbb{Z}$ and $\mathbb{Q}$ but if both axioms are removed from Zermelo-Fraenkel set theory, then no infinite Cartesian products can be proved to exist and thus the existence of $\mathbb{Z}$ and $\mathbb{Q}$ cannot be proved.

KEYWORDS: power set axiom, replacement axiom

INTRODUCTION

The earliest axiom system for set theory was invented by Zermelo in 1908. The Power Set Axiom (Table 1), which asserts the existence of the power set of any set, is one of Zermelo's axioms[1,2].

The Replacement Axiom was not part of Zermelo's system. It was independently discovered by Fraenkel and Skolem in 1922 in order to develop ordinal arithmetic and transfinite induction[2,3]. Actually, it is an axiom schema which states that if F is a function class whose domain is a set, then its range is also a set.

It has been shown that (1) the Power Set Axiom is independent from Replacement, Extensionality, Union, and Choice and (2) the Replacement Axiom is independent from Power Set, Extensionality, Union, and Choice[4].

It is also well known that the Power Set Axiom is needed for the existence of uncountable sets[5] and the Replacement Axiom is not necessary for ordinary mathematics. Therefore the existence of $\mathbb{R}$ cannot be proved without the Power Set Axiom while the Replacement Axiom is not needed for the existence of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

In this paper, we show that without the Power Set Axiom, the Replacement Axiom is sufficient for the existence of $\mathbb{Z}$ and $\mathbb{Q}$ but if both axioms are removed from Zermelo-Fraenkel (ZF) set theory, then no infinite Cartesian products can be proved to exist and thus the existence of $\mathbb{Z}$ and $\mathbb{Q}$ cannot be proved.

Table 1 The axioms of Zermelo-Fraenkel set theory.

Existence There exists a set.

Extensionality If two sets have exactly the same members, then they are equal.

Foundation Every nonempty set has an $\in$-minimal element.

Comprehension For each formula $\varphi(x, w_1, \dots, w_n)$ and any sets $z, w_1, \dots, w_n$, there exists a set y which contains exactly all those sets x in z which satisfy $\varphi(x, w_1, \dots, w_n)$.

Pairing For any sets x and y, there is a set that contains both x and y.

Union For any set $\mathcal{F}$, there is a set which contains all members of some members of $\mathcal{F}$.

Replacement For each formula $\varphi(x, y, w_1, \dots, w_n)$ and any sets $A, w_1, \dots, w_n$, if for each $x \in A$, there exists a unique y such that $\varphi(x, y, w_1, \dots, w_n)$, then there exists a set containing all y such that $\varphi(x, y, w_1, \dots, w_n)$ for some $x \in A$.

Infinity There exists an inductive set.

Power Set For any set x, there is a set containing all subsets of x.

PRELIMINARIES

This section gives some background in the well-founded sets, absoluteness, and consistency proofs which are needed for proving the main result. All basic concepts in set theory used in this paper are defined in the usual ways. We use $a, b, c, \dots, A, B, C, \dots, \mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ for sets, $\alpha, \beta, \gamma, \dots$ for ordinals, and $\varphi, \chi, \psi, \dots$ for formulae in the language of set theory. We sometimes write φ

as $\varphi(x_1,\dots,x_n)$ if the free variables of φ are among $x_1,\dots,x_n$. We write $\mathbf{ON}$ for the class of all ordinals, α^+ for the successor of α (i.e., $\alpha\cup\{\alpha\}$), $\langle x,y\rangle$ for the ordered pair of x and y, and $\mathcal{P}(x)$ for the power set of x. If T is a set of formulae, we write $T\vdash\varphi$ for T proves φ.

Z is ZF without *Replacement*. If T is a subtheory of ZF, the subtheories T−P, T$^-$, and T−Inf consist of every axiom in T except *Power Set*, *Foundation*, and *Infinity*, respectively.

A brief explanation of the notions used here will be given in an informal way. Full definitions of the concepts and proofs of all lemmas in this section can be found in Ref. 5.

A class $\mathbf{M}$ is *transitive* if every member of $\mathbf{M}$ is also a subset of $\mathbf{M}$. The *relativization* of a formula φ to a class $\mathbf{M}$, denoted by $\varphi^{\mathbf{M}}$, means φ is true in $\mathbf{M}$. For example, $(\forall y\exists x(x\neq y))^{\mathbf{M}}$ is the formula $\forall y\in\mathbf{M}\exists x\in\mathbf{M}(x\neq y)$ which means "for every y in $\mathbf{M}$, there exists an x in $\mathbf{M}$ which is distinct from y". In this case, $(\forall y\exists x(x\neq y))^{\mathbf{M}}$ if and only if $\mathbf{M}$ does not contain exactly one element.

A *sentence* is a formula which has no free variables. We say $\mathbf{M}$ is a *model* for T, where T is a set of sentences, if φ is true in $\mathbf{M}$ for all φ in T.

The well-founded sets

The definition of the class $\mathbf{WF}$ of well-founded sets is given below.

Definition 1 By transfinite recursion, we define $R(\alpha)$ by (i) $R(0)=0$, (ii) $R(\alpha^+)=\mathcal{P}(R(\alpha))$, (iii) $R(\alpha)=\bigcup_{\xi<\alpha}R(\xi)$ when α is a limit ordinal. Let $\mathbf{WF}=\bigcup_{\alpha\in\mathbf{ON}}R(\alpha)$.

Lemma 1 *Every member of $R(\omega)$ is finite.*

Lemma 2 *The Axiom of Foundation is true in any class $\mathbf{M}\subseteq\mathbf{WF}$.*

Lemma 3 *If $\mathbf{M}$ is transitive, Extensionality holds in $\mathbf{M}$.*

Lemma 4 *If $\mathbf{M}$ is a transitive model for Z$^-$−P−Inf and $\omega\in\mathbf{M}$, then the Axiom of Infinity is true in $\mathbf{M}$.*

Absoluteness

We say φ is *absolute* for $\mathbf{M}$ when φ is true in $\mathbf{M}$ if and only if φ is true. To be precise, φ is absolute for $\mathbf{M}$ if $\forall x_1,\dots,x_n\in\mathbf{M}(\varphi^{\mathbf{M}}(x_1,\dots,x_n)\leftrightarrow\varphi(x_1,\dots,x_n))$.

For example, let φ be the formula $\exists!y\forall x(x\notin y)$, $\mathbf{M}=\{1,\{1\}\}$, and $\mathbf{N}=\{1,\{2\}\}$. Since 1 is the only member in $\mathbf{M}$ such that $\forall x\in\mathbf{M}(x\notin 1)$, φ is true in $\mathbf{M}$. Since $\forall x\in\mathbf{N}(x\notin 1)$ and $\forall x\in\mathbf{N}(x\notin\{2\})$, φ is not true in $\mathbf{N}$. Since 0 is the unique set satisfying $\forall x(x\notin 0)$, φ is true. Hence φ is absolute for $\mathbf{M}$ but not for $\mathbf{N}$.

The concept of absoluteness of formulae can be extended to absoluteness of some defined notions. A defined set A is *absolute* for $\mathbf{M}$ if it exists uniquely in $\mathbf{M}$, denoted by $A^{\mathbf{M}}$, and is exactly A i.e., $A=A^{\mathbf{M}}$. Hence, from the above example, 1 is 0 in $\mathbf{M}$ i.e., $1=0^{\mathbf{M}}$. But 0 in $\mathbf{N}$ is not defined since both members of $\mathbf{N}$ satisfy the condition, so the uniqueness of the concept fails in $\mathbf{N}$. Thus we can conclude that 0 is not absolute for either $\mathbf{M}$ or $\mathbf{N}$. It is easy to see that 0 is absolute for $\{0,2\}$.

Lemma 5 *Absolute notions are closed under composition.*

Lemma 6 *The following are absolute for any transitive model for Z$^-$−P−Inf: (i) $x\in y$, (ii) $\langle x,y\rangle$, (iii) domain of f, (iv) range of f, (v) f is a one-to-one function.*

Lemma 7 *ω is absolute for any transitive model for Z−P.*

The following corollary easily follows from Lemmas 5-7.

Corollary 1 *Let $\mathbf{M}$ be a transitive model for Z−P. If every finite subset of $\mathbf{M}$ is in $\mathbf{M}$, then "x is finite" is absolute for $\mathbf{M}$. If $A\times B\in\mathbf{M}$, then $A\times B$ is absolute for $\mathbf{M}$.*

Consistency proofs

A set of formulae T is *consistent* if we cannot derive a contradiction from T.

Lemma 8 *Let T and T' be sets of sentences and $\mathbf{M}$ be a class. Suppose we can prove from T that $\mathbf{M}\neq\emptyset$ and $\mathbf{M}$ is a model for T'. Then if T is consistent, so is T'.*

The following lemma can be found in any elementary mathematical logic textbook.

Lemma 9 *For any set of formulae T and any formula φ, $T\cup\{\neg\varphi\}$ is consistent if and only if $T\nvdash\varphi$.*

Suppose T is a subtheory of ZF. Under the assumption that ZF is consistent, the above lemmas tell us that we can show that $T\nvdash\varphi$ by constructing a nonempty model for $T\cup\{\neg\varphi\}$ under ZF.

$\mathbb{Z}$ AND $\mathbb{Q}$ WITHOUT THE POWER SET AXIOM

With the Replacement Axiom

First, let us recall the definitions of $\mathbb{Z}$ and $\mathbb{Q}$.

Definition 2 The set $\mathbb{Z}$ of all integers is the quotient set $(\omega \times \omega)/\sim$ where $\sim$ is defined by

$$\langle m, n\rangle \sim \langle p, q\rangle \leftrightarrow m + q = p + n.$$

Definition 3 Addition $+_{\mathbb{Z}}$ and multiplication $\cdot_{\mathbb{Z}}$ on $\mathbb{Z}$ are defined by

$$[\langle m, n\rangle]_\sim +_{\mathbb{Z}} [\langle p, q\rangle]_\sim = [\langle m + p, n + q\rangle]_\sim$$

$$[\langle m, n\rangle]_\sim \cdot_{\mathbb{Z}} [\langle p, q\rangle]_\sim = [\langle (m \cdot p) + (n \cdot q), (m \cdot q) + (n \cdot p)\rangle]_\sim.$$

Definition 4 The set $\mathbb{Q}$ of all rationals is the quotient set $\mathbb{Z} \times (\mathbb{Z} - \{0\})/\backsim$ where $\backsim$ is defined by

$$\langle m, n\rangle \backsim \langle p, q\rangle \leftrightarrow m \cdot_{\mathbb{Z}} q = p \cdot_{\mathbb{Z}} n.$$

In general, all the notions used in the above definitions are defined in Z^- (see Ref. 6). The only use of the Power Set Axiom is in showing the existence of Cartesian products and quotient sets. Actually they can be shown to exist by using the Replacement Axiom instead of the Power Set Axiom[5]. As a result, it follows that $\mathbb{Z}$ and $\mathbb{Q}$ exist in $\mathrm{ZF}^- - \mathrm{P}$.

Without the Replacement Axiom

From the above definitions, we can see that $\mathbb{Z}$ is constructed from $\omega \times \omega$ and $\mathbb{Q}$ is constructed from $\mathbb{Z}$. In order to show that without both the Replacement Axiom and the Power Set Axiom, the existence of $\mathbb{Z}$ and $\mathbb{Q}$ cannot be proved, we will construct a model for Z$-$P under ZF which contains ω but not $\omega \times \omega$. In fact, our model contains no infinite Cartesian products.

Definition 5 Define

$$G(A) = \{\textstyle\bigcup x : x \in A\} \cup \{x \subseteq A : x \text{ is finite}\} \cup A.$$

Definition 6 By recursion, define $K(n)$ for $n \in \omega$ by $K(0) = R(\omega) \cup \mathcal{P}(\omega)$ and $K(n+1) = G(K(n))$ for all $n \in \omega$. Let $\mathcal{K} = \bigcup_{n\in\omega} K(n)$.

The following lemma follows straightforwardly from the above definition.

Lemma 10 *(i) $K(n) \subseteq K(n+1)$ for all $n \in \omega$. (ii) If $x, y \in \mathcal{K}$, then so are $\{x, y\}$ and $\bigcup x$. (iii) If A is transitive, $G(A) \subseteq \mathcal{P}(A)$. (iv) Every finite subset of $\mathcal{K}$ is in $\mathcal{K}$.*

Lemma 11 *$\forall n \in \omega$, $K(n)$ is transitive.*

Proof: The proof is by induction. For the base step, use the fact that $R(\omega)$ and $\mathcal{P}(\omega)$ are transitive. □

Corollary 2 *$\mathcal{K}$ is transitive.*

Lemma 12 $\mathcal{K} \subseteq \mathbf{WF}$.

Proof: Claim that $\forall n \in \omega$, $K(n) \subseteq R(\omega + n + 1)$. The proof proceeds by induction. Clearly, $K(0) = R(\omega) \cup \mathcal{P}(\omega) \subseteq R(\omega + 1)$. If $n = m + 1$, by Lemmas 10 (iii) and 11 and the induction hypothesis, $K(n) = G(K(m)) \subseteq \mathcal{P}(K(m)) \subseteq \mathcal{P}(R(\omega + m + 1)) = R(\omega + m + 2) = R(\omega + n + 1)$. Thus $\mathcal{K} = \bigcup_{n\in\omega} K(n) \subseteq \bigcup_{n\in\omega} R(\omega + n + 1) = R(\omega + \omega) \subseteq \mathbf{WF}$. □

Lemma 13 *If $b \in \mathcal{K}$, $b - \omega$ is finite.*

Proof: We will prove the lemma by induction on n. Let $b \in K(n)$ for some $n \in \omega$ such that b is infinite. If $n = 0$, since every member of $R(\omega)$ is finite, $b \in \mathcal{P}(\omega)$ and so $b - \omega = \emptyset$. Assume $n = m + 1$. It follows from the induction hypothesis if $b \in K(m)$. The remaining case is the case $b = \bigcup a$ for some $a \in K(m)$. Since $K(m)$ is transitive, $a \subseteq K(m)$. By the induction hypothesis, $a - \omega$ and $x - \omega$ are finite for all $x \in a$. Since $\bigcup a - \omega \subseteq \bigcup\{x - \omega \mid x \in a - \omega\}$, $\bigcup a - \omega$ is finite. □

Corollary 3 *$\mathcal{K}$ does not contain any infinite Cartesian product.*

Proof: Since every natural number is not an ordered pair, for any A and B, $A \times B = (A \times B) - \omega$. Hence if $A \times B \in \mathcal{K}$, then $A \times B$ is finite by Lemma 13. □

Corollary 4 *If $z \in \mathcal{K}$ and $y \subseteq z$, then $y \in \mathcal{K}$.*

Proof: Let $z \in \mathcal{K}$ and $y \subseteq z$. By Lemma 13, $z - \omega$ is a finite subset of $\mathcal{K}$, and so is $y - \omega$. By Lemma 10 (iv), $y - \omega \in \mathcal{K}$. Since $y \cap \omega \in \mathcal{P}(\omega) \subseteq \mathcal{K}$, by Lemma 10 (ii), $y = (y - \omega) \cup (y \cap \omega) \in \mathcal{K}$. □

Theorem 1 *$\mathcal{K}$ is a model for Z$-$P.*

Proof: Since $\mathcal{K} \neq \emptyset$, *Existence* holds in $\mathcal{K}$. By Lemmas 2, 3, 10 (ii), and 12, and Corollary 2, *Extensionality*, *Foundation*, *Pairing*, and *Union* hold in $\mathcal{K}$. By Corollary 4, *Comprehension* holds in $\mathcal{K}$. Finally, $\mathcal{K}$ satisfies *Infinity* by Lemma 4. □

Lemma 14 *"x is finite" and "x is a Cartesian product" are absolute for $\mathcal{K}$.*

Proof: By Corollaries 1 (i) and 2, Lemma 10 (iv), and Theorem 1, "x is finite" is absolute for $\mathcal{K}$.

Let $\phi(x)$ be the formula $\exists A \exists B (x = A \times B)$. Then x is a Cartesian product if and only if $\phi(x)$. Suppose $x \in \mathcal{K}$ and $\phi(x)$ i.e., $x = A \times B \in \mathcal{K}$. Since $\mathcal{K}$ satisfies *Union*, A, $B \subseteq \bigcup\bigcup x \in \mathcal{K}$. By Corollary 4, A, $B \in \mathcal{K}$, and so $A \times B$ is absolute for $\mathcal{K}$ by Corollary 1 (ii). We have $\phi(x)^{\mathcal{K}}$. Thus $\phi(x) \to \phi(x)^{\mathcal{K}}$. The converse is obvious. □

Theorem 2 *$\mathcal{K}$ is a model for $\forall x$ (x is not an infinite Cartesian product).*

Proof: Follows by Corollary 3 and Lemma 14. □

Corollary 5 *If ZF is consistent, then Z−P$\nvdash$ $\exists x$ (x is an infinite Cartesian product).*

Proof: Follows by Lemmas 8 and 9 and Theorems 1 and 2. □

Corollary 6 *If ZF is consistent, then Z−P$\nvdash$ $\mathbb{Z}$ exists.*

Proof: The proof is by contraposition. Suppose Z−P $\vdash \mathbb{Z}$ exists. Since $\bigcup \mathbb{Z} = \omega \times \omega$, Z−P $\vdash \omega \times \omega$ exists.

Let S be the successor function. Since $S \subseteq \omega \times \omega$, Z−P $\vdash$ S exists. Since the Pigeonhole Principle can be proved in Z−P[6] and S is a bijection from ω onto $\omega - \{0\}$, Z−P $\vdash$ ω is infinite, and so Z−P $\vdash$ $\omega \times \omega$ is infinite. By Corollary 5, the proof is complete. □

Corollary 7 *If ZF is consistent, then Z−P$\nvdash$ $\mathbb{Q}$ exists.*

Proof: Follows from the definition of $\mathbb{Q}$ and Corollary 6. □

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