RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2008.34.239

ScienceAsia 34 (2008): **239–244**

Superconvergence of iterated numerical solutions using wavelets

Supot Seebut[a] and Boriboon Novaprateep[b,*]

[a] Institute for Innovation and Development of Learning Process, Mahidol University, Rama 6 Road, Bangkok 10400, Thailand

[b] Department of Mathematics, Faculty of Science, Mahidol University, Rama 6 Road, Bangkok 10400, Thailand

[*] Corresponding author, E-mail: scbnv@mahidol.ac.th

Received 12 Mar 2007
Accepted 2 Jan 2008

ABSTRACT: In this paper, we examine the superconvergence property of iterates of numerical solutions to both Fredholm integral equations of the second kind and to nonlinear Hammerstein equations. The iterates are obtained by applying a class of multiwavelets developed by Alpert.

KEYWORDS: iterated degenerate kernel method, wavelet degenerate kernel method, Hammerstein equations

INTRODUCTION

In this paper, we present the results on superconvergence of the iterated variants of the numerical solutions obtained by the wavelet degenerate kernel method. The use of wavelets in the degenerate kernel method plays a critical role in obtaining this superconvergence which is the convergence that converges faster than generally expected. Superconvergence of the iterated variants for the Galerkin method as well as the collocation method has been studied extensively in recent years[1–3]. We remark that, as the method of Alpert is the Galerkin method, we expect that there would be a superconvergence when this solution is iterated. In this paper, we take advantage of the orthonormality of the wavelet basis, resulting in a least-squares approximation of the kernel, to obtain the superconvergence in the L_2 norm of the wavelet degenerate kernel method. Fredholm equations of the second kind with smooth as well as weakly singular kernels will be treated. Finally, the results obtained in relation to the Fredholm equations will be extended to a class of nonlinear Hammerstein equations with kernels having similar characteristics.

Multiwavelet bases and Fredholm equations

Making use of the multiresolution analysis developed by Mallet[4] and Meyer[5], Alpert[6] constructed the following multiwavelet bases for $L_2[0,1]$. For a positive integer k, and for $m \in \mathbb{N}$ (where $\mathbb{N} = \{0,1,2,\ldots\}$) we define the space S_m^k of piecewise polynomials by $S_m^k = \{f$: the restriction of f to the interval $(2^{-m}n, 2^{-m}(n+1))$ is a polynomial of degree less than k, for $n = 0,\ldots,2^m-1$, and f vanishes elsewhere$\}$. We have $S_m^k \in S_{m+1}^k$ and the dimension of S_m^k is $2^m k$. For $m \in \mathbb{N}$, we denote by R_m^k the orthogonal complement of S_m^k in S_{m+1}^k:

$$S_m^k \oplus R_m^k = S_{m+1}^k, \quad R_m^k \perp S_m^k.$$

It is clear that

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k. \tag{1}$$

Now if we find an orthogonal basis $\{h_i\}_{i=1}^k$ for R_0^k, then since $R_0^k \perp S_0^k$, the first k moments of $h_1,\ldots,h_k$ vanish:

$$\int_0^1 h_j(x)x^i\,\mathrm{d}x = 0, \quad i = 0,1,2,\ldots,k-1.$$

The wavelet basis of Alpert is constructed by defining orthogonal systems

$$h_{j,m}^k(x) = 2^{m/2}h_j(2^m x - n), \quad j = 1,\ldots,k.$$

The detailed construction of $h_1,\ldots,h_k$ is found in Ref. 6. From this, we obtain

$$R_0^k = \text{linear span}\{h_i : i = 1,\ldots,k\},$$

and, in general,

$$R_m^k = \text{linear span}\{h_{j,m}^n : n = 0,\ldots,2^m-1\}. \tag{2}$$

We define space S^k by

$$S^k = \bigcup_{m=0}^{\infty} S_m^k, \tag{3}$$

so that S^k is dense in $L_2[0,1]$. Moreover, if $\{u_1,\dots,u_k\}$ is an orthonormal basis for S_0^k, then it follows from (1)–(3) that

$$B_k \equiv \{u_j\} \cup \{h_{j,m}^n\} \tag{4}$$

in which $j = 1,\dots,k$, $m \in \mathbb{N}$, $n = 0,\dots,2^m-1$, form an orthonormal system for $L_2[0,1]$. B_k is referred to as the multiwavelet basis of order k for $L_2[0,1]$. For each positive integer k and $m \in \mathbb{N}$, the orthogonal projection $Q_m^k f$ of $f \in L_2[0,1]$ onto S_m^k is given by

$$(Q_m^k f)(x) = \sum_{j=1}^{k} \sum_{n=0}^{2^m-1} \langle f, u_{j,m}^n \rangle\, u_{j,m}^n(x), \tag{5}$$

where $\{u_{j,m}^n\}$ is an orthonormal basis for S_m^k. Here we assume that the bases for $S_0^k, R_0^k, \dots, R_{m-1}^k$ are rearranged to form the basis $\{u_{j,m}^n\}$. The power of approximation of the multiwavelets is given in the following theorem[6].

Theorem 1 *Suppose that the function* $f : [0,1] \to R$ *is* k *times continuously differentiable,* $f \in C^k[0,1]$. *Then*

$$\left\|Q_m^k f - f\right\|_2 \leqslant 2^{-mk} \frac{2}{4^k k!} \sup_{x\in[0,1]} |f^{(k)}(x)|.$$

We are now interested in applying the multiwavelet basis for $L_2[0,1]$ to obtain approximate solutions of the Fredholm integral equation of the second kind. The equation can be written as

$$f(x) - \int_0^1 \kappa(x,t) f(t)\,\mathrm{d}t = g(x), \quad x \in [0,1] \tag{6}$$

or

$$f - Kf = g \tag{7}$$

where

$$(Kf)(x) = \int_0^1 \kappa(x,t) f(t)\,\mathrm{d}t, \quad x \in [0,1],$$

for $\kappa \in L_2([0,1]\times[0,1])$, $g \in L_2[0,1]$ and f is the function to be determined for all $f \in L_2[0,1]$. As stated in the introduction, an application of the multiwavelet basis to the integral equation produces a linear system consisting of a sparse matrix.

There are several numerical methods that one can choose to approximate the solution of (6). The Galerkin method and the collocation method are two of the most widely used numerical schemes[7,8]. On the other hand, the degenerate kernel method consists of approximating the kernel κ in the integral operator K of (7) by a tensor product of univariate functions. In particular, we are interested in using multiwavelets for these univariate functions. Let $\{b_1, b_2, \dots\}$ denote the orthonormal basis for $L_2[0,1]$ comprising of the multiwavelet elements (see (4)). We remark that the statements in this and the next section concerning the superconvergence of the iterated degenerate kernel method remains valid as long as the set $\{b_1, b_2, \dots\}$ is an orthonormal basis. More on this point can be found in Ref. 9. However, the sparsity of the resulting matrix is normally lost without the assumption of wavelets with vanishing moments.

The use of the degenerate kernel method has not been widely recognized in solving the Fredholm equations of the second kind because of its higher computational cost compared with the collocation method. However, the use of a wavelet basis makes the method more attractive. Moreover, the degenerate kernel method provides an error estimate that depends on the order to which the kernel is approximated, while the collocation method as well as the Galerkin method requires certain smoothness conditions of the solution to guarantee the optimal convergence rate. A priori error estimation of this type is difficult to handle as one usually has no knowledge about the solution in the case of many practical problems. We assume that the b_i's are enumerated so that $\{b_i\}_{i=1}^{2^m k}$ forms the basis for S_m^k, (see (5)). Now for $\kappa \in L_2([0,1]\times[0,1])$, we have

$$\kappa(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \kappa_{ij} b_i(x) b_j(t), \tag{8}$$

where

$$\kappa_{ij} = \int_0^1 \int_0^1 \kappa(x,t) b_i(x) b_j(t)\,\mathrm{d}x\,\mathrm{d}t, \quad i,j = 1,2,\dots.$$

We approximate the integral operator K by the finite rank operator K_n defined by

$$(K_n f)(x) = \int_0^1 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\kappa_{ij} b_i(x) b_j(t)\right) f(t)\,\mathrm{d}t, \tag{9}$$

with $x \in [0,1]$, $f \in L_2[0,1]$ and $n = 2^m k$. Clearly,

$$\left\|K - K_n\right\|_2 \to 0 \quad \text{as } n \to \infty.$$

An approximate solution is found by solving

$$f_n - K_n f_n = g. \tag{10}$$

Specifically, to solve (10) for f_n, notice that (10) can be written as

$$f_n(x) - \sum_{i=1}^{n} b_i(x) \int_0^1 \sum_{j=1}^{n} \kappa_{ij} b_j(t) f_n(t)\,\mathrm{d}t = g(x).$$

Hence, if we put

$$c_i \equiv \int_0^1 \sum_{j=1}^{n} \kappa_{ij} b_j(t) f_n(t)\,\mathrm{d}t, \tag{11}$$

then

$$f_n(x) = g(x) + \sum_{i=1}^{n} c_i b_i(x). \tag{12}$$

Substituting the form of f_n in (12) into (11), we obtain

$$c_i - \sum_{j=1}^{n} c_j \kappa_{ij} = \int_0^1 \sum_{j=1}^{n} \kappa_{ij} b_j(t) g_n(t)\,\mathrm{d}t, \tag{13}$$

for each $i = 1, 2, \ldots, n$. The system (13) may be solved numerically for c_i from which we obtain f_n using (12). Assuming that $(I-K)^{-1}$ exists, where I is the identity operator, we obtain the error estimate,

$$\begin{aligned} f - f_n &= Kf - K_n f_n \\ &= (K - K_n)f + K_n(f - f_n) \end{aligned}$$

and hence

$$f - f_n = (I - K_n)^{-1}(K - K_n)f.$$

From this it follows that

$$\|f - f_n\|_2 \leqslant \|(I-K)^{-1}\|_2 \|K - K_n\|_2 \|f\|_2. \tag{14}$$

By Theorem 1, $\|Kf - K_n f\|_2 = O(2^{-mk})$ and therefore

$$\|f - f_n\|_2 = O(2^{-mk}). \tag{15}$$

Observe that the rate of convergence of the relative error of the approximation depends upon the order of approximation of K by K_n, but not upon the smoothness of f (see (14)). This is not the case for the projection methods that include the collocation and Galerkin methods as special cases[7].

What Alpert observed at this point is that a large majority of κ_{ij} can be neglected, resulting in a sparse matrix for the linear system. This observation carries over to the present method, as the matrix of the Galerkin method and that of the degenerate kernel method coincide. More precisely, he defines the notion of the separation from the diagonal of a support of $b_i \otimes b_j$ as follows:

Definition 1 *We say that a rectangular region $[t, t+a] \times [s, s+b] \subset R^2$ is separated from the diagonal if $a + \max(a,b) \leqslant s - t$ or $b + \max(a,b) \leqslant t - s$.*

For the logarithmic kernel $\kappa(t,s) = \log|t-s|$, the importance of this definition is manifested in the boundary integral formulation of Laplace's equation. We state Lemma 2.2 from Ref. 6:

Lemma 1 *Suppose that $\kappa(t,s) = \log|t-s|$ is given and B_k is the multiwavelet basis of order k for $L_2[0,1]$. Denote the supports of $b_i(t)$ and $b_j(s)$ in B_k by $[x_0, x_0+a]$ and $I_j = [y_0, y_0+b]$, respectively, and assume that they are separated from the diagonal. Then*

$$|\kappa_{ij}| \leqslant \frac{\sqrt{ab}}{2k \cdot 3^{k-1}}. \tag{16}$$

For a smooth kernel, we have the following (Lemma 2.31 from Ref. 6):

Lemma 2 *Suppose that $\kappa(t,s) = f(t,s)\log|t-s| + g(t,s)$ and $\kappa(t,s) : D \times D \to C$ where D is the closed disc of radius $\frac{3}{2}$ centred at $t = \frac{1}{2}$. Also, f and g are analytic in a domain containing $D \times D \to C^2$ and B_k is the multiwavelet basis of order k for $L_2[0,1]$. Denote the supports of $b_i(t)$ and $b_j(s)$ in B_k by $I_i = [x_0, x_0+a]$ and $I_j = [y_0, y_0+b]$, respectively, and assume that they are separated from the diagonal. Then*

$$|\kappa_{ij}| \leqslant \left(\frac{k}{8} + \frac{3}{16}\right) \frac{1}{3^{k-1}} \sup_{t,s \in \partial D} |f(t,s)| + \frac{\sqrt{ab}}{7 \cdot 8^k} \sup_{t,s \in \partial D} |g(t,s)|.$$

Regarding the number of the basis elements which have their supports near the diagonal, we recall Lemma 2.41 of Ref. 6:

Lemma 3 *Suppose that $I_1, I_2, \ldots, I_n$ are the non-increasing intervals of support of the first n functions of the basis B_k. Of the n^2 rectangular regions $I_i \times I_j$, we denote the number separated from the diagonal by $S(n)$ and the number near the diagonal by $N(n) = n^2 - S(n)$. Then $N(n) = O(n \log n)$.*

We now define the iterated variants of f_n by

$$f_n^{\mathrm{I}} = g + K f_n. \tag{17}$$

In the case that a numerical solution of (6) is obtained by the Galerkin or collocation method, it is well known that under suitable conditions the corresponding iterated variants converge to f in the L_∞ norm at a faster rate than the original numerical solution converges to f. With a sufficiently smooth kernel[1,10], the rate of convergence of the iterated variants is twice as fast as that of the original convergence, a phenomenon commonly known as superconvergence. We point out that it is somewhat more involved to get a superconvergence of the iterated collocation variants than that of the iterated Galerkin method[1,2]. In the case of Fredholm integral equations of the second kind with weakly singular kernels, a certain enhancement in the convergence rate was also obtained for the iterated variants[3]. We also remark that many results on the superconvergence of the iterates for the Fredholm equations have been generalized to a class of nonlinear Hammerstein equations[2,3]. In this paper, we obtain a superconvergence of the iterated degenerate kernel method using the wavelets of Alpert. The success of the method hinges heavily upon the orthonormality of the wavelet basis. Namely, we show that the best L_2 approximation for the kernel κ plays an important role. In the next section, we consider the Fredholm equations of the second kind, and in the final section, we consider a class of nonlinear Hammerstein equations.

THE ITERATED VARIANTS FOR FREDHOLM INTEGRAL EQUATIONS

We let $W_2^k = W_2^k[0,1]$, where k is a non-negative integer, denote the Sobolev space of functions defined over $[0,1]$. In other words, $f \in W_2^k$ if and only if $f^{(j)} \in L_2 = L_2[0,1]$ for $j = 0,1,\ldots,k$, where $f^{(j)}$ denotes the jth distributional derivative of f. The space W_2^k is equipped with the norm

$$\|f\|_{k,2} = \sum_{j=0}^{k} \|f^{(j)}\|_2.$$

The Sobolev space $W_2^k([0,1]\times[0,1])$ of bivariate functions is defined similarly: $f \in W_2^k([0,1]\times[0,1])$ if $\partial^{(l)} f/\partial^{(i)}x\partial^{(j)}t \in L_2([0,1]\times[0,1])$ for $i+j=l$ and $l = 0,1,\ldots,k$. Theorem 1 can be easily extended to obtain an L_2 estimate of $Q_m^k f$ for $f \in W_2^k$.

Theorem 2 *Suppose $f \in W_2^k$. Then $Q_m^k f$ of (5) approximates f in the L_2 norm as follows:*

$$\|Q_m^k f - f\|_2 \leqslant C 2^{-mk} \|f^{(k)}\|_2, \tag{18}$$

where C is a constant independent of m and k.

Proof: Recall that S_m^k denotes the space of splines of degree k with knots at $2^{-m}n$, $n = 0,1,\ldots,2^m-1$. Then it is well known (see p. 230 of Ref. 11) that

$$\inf_{g\in S_m^k} \|f-g\|_2 \leqslant C 2^{-mk} \|f^{(k)}\|_2.$$

Now, noting that $Q_m^k f$ is the best L_2 approximation of f in S_m^k, we obtain the desired result. □

From (8), (9), and the orthonormality of $\{b_1, b_2, \ldots\}$, the kernel defined in (9) is the least-squares approximation of $\kappa(x,t)$:

$$\int_0^1\!\!\int_0^1 \left|\kappa(x,t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \kappa_{ij} b_i(x) b_j(t)\right|^2 \mathrm{d}x\,\mathrm{d}t = \min_{a_{ij}\in R} \int_0^1\!\!\int_0^1 \left|\kappa(x,t) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_i(x) b_j(t)\right|^2 \mathrm{d}x\,\mathrm{d}t.$$

We are now ready to state and prove the main theorem of this section, which provides an estimate for the order of convergence of the iterated variants.

Theorem 3 *Let f and f_n be the solutions of (7) and (10), respectively. Assume that $f \in W_2^k([0,1])$, $\kappa(x,t) \in W_2^k([0,1]\times[0,1])$ and $\psi_x(u,s) \equiv \kappa(x,u) f_n(s) \in W_2^k([0,1]\times[0,1])$ for each $x \in [0,1]$, where $0 < l < k$ and that $\|\psi_x\|_2$ is uniformly bounded for $x \in [0,1]$. Then*

$$\|f - f_n^{\mathrm{I}}\|_2 = O(2^{-2mk}).$$

Proof: Note first that

$$f - f_n^{\mathrm{I}} = K(f - f_n). \tag{19}$$

Using $Kf = Kg + K^2 f$ and $Kf_n = KK_n f_n + Kg$, we obtain

$$K\hat{f}_n = K_n(Kf - Kf_n) + \hat{K}_n(Kf - Kf_n) + K\hat{K}_n f_n$$

in which we have introduced the notation $\hat{K}_n \equiv K - K_n$, $\hat{f}_n \equiv f - f_n$. Since $\|\hat{K}_n\|_2 \to 0$ as $n \to \infty$ and $(I-K)^{-1}$ exists by assumption, we conclude (following Ref. 7) that $(I-K)^{-1}$ exists and is uniformly bounded for sufficiently large n. Therefore,

$$K\hat{f}_n = (I - K_n)^{-1}\{\hat{K}_n(Kf - Kf_n) + K\hat{K}_n f_n\}.$$

Taking the norm on both sides,

$$\|K\hat{f}_n\|_2 \leqslant \{\|\hat{K}_n\|_2 \|K\|_2 \|\hat{f}_n\|_2 + \|K\hat{K}_n f_n\|_2\} \times \|(I-K_n)^{-1}\|_2. \tag{20}$$

Since

$$\hat{f}_n = (I - K_n)^{-1} \hat{K}_n f, \tag{21}$$

and from equations (19)–(21),

$$\begin{aligned} \|f - f_n^{\mathrm{I}}\|_2 &= \|K\hat{f}_n\|_2 \\ &\leqslant C\{\|\hat{K}_n\|_2 \|\hat{f}_n\|_2 + \|K\hat{K}_n f_n\|_2\} \\ &\leqslant C\{\|\hat{K}_n\|_2^2 + \|K\hat{K}_n f_n\|_2\}. \end{aligned} \tag{22}$$

From (22), using Theorem 2, we obtain

$$\left\|f - f_n^{\mathrm{I}}\right\|_2 = O(2^{-2km}) + O(\left\|K\hat{K}_n f_n\right\|_2). \quad (23)$$

It remains to estimate the order of convergence of $\|K\hat{K}_n f_n\|_2$. We note that $\|K\hat{K}_n f_n\|_2 \leqslant \|K\hat{K}_n f_n\|_\infty$ for some $C > 0$, and that

$$\begin{aligned}|K\hat{K}_n f_n(t)| &= \left|\int_0^1 \kappa(t,u) \int_0^1 \hat{\kappa}_n(u,s) f_n(s)\,\mathrm{d}s\,\mathrm{d}u\right| \\ &= \left|\int_0^1\!\!\int_0^1 \kappa(t,u)\hat{\kappa}_n(u,s) f_n(s)\,\mathrm{d}s\,\mathrm{d}u\right|\end{aligned}$$

where $\hat{\kappa}_n(u,s) \equiv \kappa(u,s) - \kappa_n(u,s)$. Let $\varphi_n(u,s) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} b_i(u) b_j(s)$ be any element that is a tensor product of multiwavelets $\{b_i\}_{i=1}^n$ with $n = 2^m k$. Then since κ_n is the least-squares approximation of κ,

$$\int_0^1\!\!\int_0^1 \varphi_n(u,s)\hat{\kappa}_n(u,s)\,\mathrm{d}s\,\mathrm{d}u = 0,$$

and therefore,

$$|K\hat{K}_n f_n(t)| = \left|\int_0^1\!\!\int_0^1 [\psi_t(u,s) - \varphi_n]\hat{\kappa}_n\,\mathrm{d}s\,\mathrm{d}u\right|.$$

Applying the Cauchy-Schwartz inequality,

$$|K\hat{K}_n f_n(t)| \leqslant \left\|\psi_t - \varphi_n\right\|_2 \left\|\hat{\kappa}_n\right\|_2.$$

In the above inequality, the second $\|\cdot\|_2$ denotes the L_2 norm defined on the space of bivariate functions $W_2^k([0,1]\times[0,1])$. Noting that $\|\hat{\kappa}_n\|_2 = O(2^{-mk})$ and selecting φ_n so that $\|\psi_t - \varphi_n\|_2 = O(2^{-mk})$ is satisfied, equation (23) proves the desired result. □

HAMMERSTEIN EQUATIONS

In this section, we generalize the results of the previous section to a class of Hammerstein equations. The Hammerstein equations arise naturally in connection with Laplace's equation in two-dimensional space having a certain type of nonlinear boundary conditions. The Hammerstein equation can be written as

$$f(x) - \int_0^1 \kappa(x,t)\psi(t,f(t))\,\mathrm{d}t = g(x), \quad 0 \leqslant x \leqslant 1. \quad (24)$$

We assume the following conditions: $\lim_{t\to\tau} \|\kappa_t - \kappa_\tau\|_\infty = 0$, $\tau \in [0,1]$, where $\kappa_a(b) \equiv \kappa(a,b)$;

$$M = \sup_{0\leqslant s\leqslant 1} \int_0^1 |\kappa(t,s)|\,\mathrm{d}t < \infty;$$

$g \in C[0,1]$; $\psi(t,s)$ is continuous in $t \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which $|\psi(t,x_1) - \psi(t,x_2)| \leqslant C_1|x_1 - x_2|$, for all $x_1, x_2 \in (-\infty,\infty)$; the partial derivative $\psi^{(a,b)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that $|\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \leqslant C_2|x_1 - x_2|$, for all $x_1, x_2 \in (-\infty,\infty)$; for $f \in C[0,1]$, we have $\psi(\cdot, f(\cdot)), \psi^{(0,1)}(\cdot, f(\cdot)) \in C[0,1]$.

The degenerate kernel method for Hammerstein equations was first established by Kaneko and Xu[12]. We define

$$K\Psi f(x) \equiv \int_0^1 \kappa(x,t)\psi(t,f(t))\,\mathrm{d}t$$

so that (24) can be written as

$$f - K\Psi f = g. \quad (25)$$

By analogy to equation (9), we approximate the operator $K\Psi$ by

$$(K_n\Psi f)(x) = \int_0^1 \sum_{i=1}^\infty \sum_{j=1}^\infty \kappa_{ij} b_i(x) b_j(t) \psi(t,f(t))\,\mathrm{d}t \quad (26)$$

with $x \in [0,1]$, where $\{b_i\}_{i=1}^\infty$ is an orthonormal basis for $L_2[0,1]$ as defined earlier. Let f_n denote the solution of the equation,

$$f_n - K_n\Psi f_n = g. \quad (27)$$

The iterated solution f_n^{I} is now defined by

$$f_n^{\mathrm{I}} = g + K\Psi f_n. \quad (28)$$

If we let

$$c_i \equiv \sum_{j=1}^n \int_0^1 \kappa_{ij} b_j(t)\psi(t,f_n(t))\,\mathrm{d}t, \quad (29)$$

f_n can be written as

$$f_n(x) = g(x) + \sum_{i=1}^n c_i b_i(x). \quad (30)$$

Substituting (30) into (29), we obtain the following n nonlinear equations in n unknowns $c_1, \ldots, c_n$,

$$c_i = \sum_{j=1}^n \int_0^1 \kappa_{ij} b_j \psi\left(t, g(t) + \sum_{l=1}^n c_l b_l(t)\right)\,\mathrm{d}t, \quad (31)$$

for $1 \leqslant i \leqslant n$. The Fréchet derivative of $K\Psi$ at $\varphi_0 \in C[a,b]$ is denoted and defined by

$$(K\Psi)'(\varphi_0)(\varphi)(x) = \int_0^1 \kappa(x,t)\psi_2(t,\varphi_0(t))\varphi(t)\,\mathrm{d}t$$

for $\varphi \in C[a,b]$, where ψ_2 denotes the partial derivative of ψ with respect to the second variable. The following theorem describes the superconvergence of f_n^{I} to f.

Theorem 4 *Assume that f and f_n are solutions of equations (24) and (27), respectively. Assume also that in (24), $\kappa(x,t), \eta_t(u,s) \in W_2^k([0,1]\times[0,1])$, where $\eta_t(u,s) \equiv \kappa_t(t,u)\psi(s,f_n(s))$. Finally assume that 1 is not an eigenvalue of $(K\Psi)'(f)$. Then $\|f - f_n^{\mathrm{I}}\|_2 = O(2^{-2mk})$.*

Proof: From (25) and (28),

$$f - f_n^{\mathrm{I}} = K\Psi f - K\Psi f_n. \tag{32}$$

Now,

$$\begin{aligned} &K\Psi f - K\Psi f_n \\ &\quad = K\Psi(g + K\Psi f) - K\Psi(g + K_n\Psi f_n) \\ &\quad = K_{\theta(n)}(K\Psi f - K_n\Psi f_n) \end{aligned}$$

where $K_{\theta(n)} \equiv (K\Psi)'\{\theta(n)(g + K_n\Psi f_n) + (1-\theta(n))(g + K\Psi f)\}$ for some $0 \leqslant \theta(n) \leqslant 1$. Since K is compact, $(K\Psi)'(f)$ is also compact. Also, since the solutions f_n of the degenerate kernel method converge to the solution f of the Hammerstein equation[12], $\{K_{\theta(n)}\}$ converges to $(K\Psi)'(f)$ in the operator norm. From this, since 1 is not an eigenvalue of $(K\Psi)'(f)$, using theorem 10.1 of Ref. 13 we show that $(I - K_{\theta(n)})^{-1}$ exists and is uniformly bounded for sufficiently large n. Hence, we obtain

$$K\Psi f - K\Psi f_n = (I - K_{\theta(n)})^{-1} K_{\theta(n)}(K - K_n)\Psi f_n. \tag{33}$$

Combining (32) and (33), and taking the norm on both sides, we obtain

$$\left\|f - f_n^{\mathrm{I}}\right\|_2 \leqslant c\left\|K_{\theta(n)}(K - K_n)\Psi f_n\right\|_2$$

for some constant c independent of n. Using the assumptions on κ and η_t and arguing as in the proof of Theorem 3, we obtain the desired result. □

ACKNOWLEDGEMENTS

This work was supported by the National Center for Genetic Engineering and Biotechnology and the Commission on Higher Education, Ministry of Education, Thailand.

REFERENCES

1. Graham IG, Joe S, Sloan IH (1985) Iterated Galerkin versus iterated collocation for integral equations of the second kind. *IMA J Numer Anal* **5**, 355–69
2. Kaneko H, Noren RD, Padilla PA (1997) Superconvergence of the iterated collocation method for Hammerstein equations. *J Comput Appl Math* **80**, 335–49.
3. Kaneko H, Xu Y (1996) Superconvergence of the iterated Galerkin methods for Hammerstein equations. *SIAM J Numer Anal* **33**, 1048–64.
4. Mallet S (1989) Multiresolution approximation and wavelet orthonormal bases of $L_2(R)$. *Trans Amer Math Soc* **315**, 69–88.
5. Meyer Y (1986) Ondelettes et functions splines. Tech. Report, Seminaire EDP, École Polytechnique, Paris.
6. Alpert BK (1993) A class of bases in L_2 for the sparse representation of integral operators. *SIAM J Math Anal* **4**, 246–62.
7. Atkinson KE (1997) The numerical solution of integral equations of the second kind. Cambridge Monographs on Applied and Computational Mathematics, Cambridge Univ. Press.
8. Delves LM, Mohamed JL (1985) Computational methods for integral equations. Cambridge Univ. Press.
9. Kaneko H, Xu Y, Padilla PA (2001) Superconvergence of the iterated degenerate kernel method. *Appl Anal* **80**, 331–51.
10. Sloan IH (1990) Superconvergence. In: Golberg MA (ed) Numerical Solutions of Integral Equations. Plenum Press, New York, pp 35–70.
11. Schumaker LL (1981) Spline functions: Basic theory. Wiley, New York.
12. Kaneko H, Xu Y (1991) Degenerate kernel method for Hammerstein equations. *Math Comp* **56**, 141–8.
13. Kress R (1989) Linear integral equations. Springer-Verlag, Heidelberg.