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CHARACTERIZING DISCRETE EXPONENTIAL POLYNOMIALS BY CASORATI'S DETERMINANTS

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ABSTRACT

For a real and sufficiently smooth function f, Karlin and Loewner proved two interesting results which say roughly that

(i) f is an exponential polynomial if and only if certain Wronskian vanishes;

(ii) f is an exponential sum with positive coefficients if a Wronskian of certain order vanishes, while those of lower orders have the same positive sign at one point.

We give a slightly different proof of (ii) and obtain analogues of both results in the discrete setting, with Casorati's determinants taking the place of Wronskians.

INTRODUCTION

A (real) exponential polynomial of <u>order</u> r is an expression of the form

$$f(x) = P_1(x)\, e^{\alpha_1 x} + \dots + P_k(x)\, e^{\alpha_k x} \qquad (x \in R),$$

where $\alpha_1, \dots, \alpha_k$ are distinct real numbers; $P_1, \dots, P_k$ are real polynomials of degrees deg P_i $(i = 1,\dots,k)$, and $\sum_{i=1}^{k} (\deg P_i + 1) = r$. Given a function f differentiable up to a sufficiently high order, define

$$H_{r+1}(f) := W(f,f',\dots,f^{(r)}) = \begin{vmatrix} f(x) & f'(x) & \dots & f^{(r)}(x) \\ f'(x) & f''(x) & \dots & f^{(r+1)}(x) \\ \dots & & & \\ f^{(r)}(x) & f^{(r+1)}(x) & \dots & f^{(2r)}(x) \end{vmatrix}$$

to be the <u>Wronskian</u> of the functions $f,f',\dots,f^{(r)}$. Karlin and Loewner [2] obtained the following interesting results concerning the problem of characterizing those functions f for which $H_r(f)$ maintains a constant sign over an interval.

I. Let $f \in C^{2r}$ on (a,b). Then f satisfies $H_{r+1}(f) \equiv 0$ if and only if f is a real exponential polynomial of order at most r.

An immediate consequence of I is the following: if $f \in C^{2r+2}$ on (a,b) and satisfies $H_{r+1}(f) = Ae^{\alpha x}$, for prescribed real constants A and α, then f is an exponential polynomial of order at most r+1.

II. If $f \in C^{2r}$ on (a,b) and satisfies $H_{r+1}(f) \equiv 0$, while at $x = x_0 \in$ (a,b) we have

$$H_1(f) > 0,\ H_2(f) > 0,\ \dots,\ H_r(f) > 0\,, \qquad (1)$$

then $f(x) = \sum_{i=1}^{r} a_i e^{\alpha_i x}$, where $a_i > 0$ and a_i are real and distinct $(i = 1,\dots,r)$, and therefore (1) holds for all real r.

III. Let $f(x) = \sum_{m=0}^{\infty} \frac{c_m x^m}{m!}$ be a power series expansion about the origin convergent for $|x| < \rho$. If f satisfies

$$\varepsilon_k \begin{vmatrix} f^{(r)}(0) & \dots & f^{(r+k)}(0) \\ f^{(r+1)}(0) & \dots & f^{(r+k+1)}(0) \\ \dots & & \\ f^{(r+k)}(0) & \dots & f^{(r+2k)}(0) \end{vmatrix} > 0 \qquad (r = 0,1,\dots\ ;\ k = 0,1,\dots,n)$$

where $\varepsilon_0, \dots, \varepsilon_k$ are prescribed sequence of sign +1 or -1, then

$$\varepsilon_k \begin{vmatrix} f^{(r)}(x) & \dots & f^{(r+k)}(x) \\ f^{(r+1)}(x) & \dots & f^{(r+k+1)}(x) \\ \dots & & \\ f^{(r+k)}(x) & \dots & f^{(r+2k)}(x) \end{vmatrix} > 0, \quad \text{for } 0 \le x < \rho;\ r = ,1,\dots\ ;\ k = 0,1,\dots,n$$

The final stage of the proof of II in [2] appeals to the uniqueness criteria of differential equations. This is indeed unnecessary, and in the next section, we give a slightly different proof of II without making such appeal. Our main objective (in Sections 3 and 4) is to derive analogues of I and II in the discrete setting, i.e. the case of difference equations. Though the central ideas of Karlin and Loewner carry over, the discrete nature makes it necessary to impose further restrictions, as well as to replace Wronskians by Casorati's determinants. Regarding the anlogue of III, this is quite trivial, and we shall merely remark upon it at the end of the paper.

A PROOF OF II

We may take $x_0 = 0$, for otherwise it suffices to consider $f(x-x_0)$ instead of f(x). By I, f is an exponential polynomial of order r, say $f(x) = \sum_{i=1}^{k} a_i e^{\alpha_i x}$, where for $i = 1,\dots,k$, the a_i are distinct real numbers, P_i real polynomials with

$$\sum_{i=1}^{k} (\deg P_i + 1) = r.$$

We claim that there exist positive constants b_i and distinct real β_i $(i=1,\dots,r)$ such that the exponential polynomial $g(x) = \sum_{i=1}^{r} b_i e^{\beta_i x}$ interpolates f(x) at the origin up to the $(2r-1)^{th}$ derivative, i.e.

$$g^{(j)}(0) = f^{(j)}(0) \qquad (j = 0,1,\dots,2r-1).$$

This amounts to solving a moment problem on the positive axis such that

$$f^{(j)}(0) = \sum_{i=1}^{r} b_i \beta_i^j \qquad (j = 0,1,\ldots,2r-1).$$

By Theorems 17 and 18, pp. 235-238 of Gantmacher [1], this problem is uniquely solvable, and the claim is verified.

It remains to show that $f(x) \equiv g(x)$. We consider the exponential polynomial $f(x) - g(x)$, which is of order at most 2r, and vanishes at the origin up to the $(2r-1)^{th}$ derivative. By a result on generalized Vandermonde determinants, see p. 283 of van der Poorten [4], we must have $f(x) - g(x) \equiv 0$, and the proof is complete.

THE FIRST MAIN THEOREM

By a discrete exponential polynomial , we mean a function

$$f(x) = \sum_{i=1}^{k} P_i(x)\alpha_i^x ,$$

where α_i are distinct real numbers, P_i polynomials with real coefficients $(i = 1,\ldots,k)$, and the variable x takes on only nonnegative integral values. It is said to be of order r if $\sum_{i=1}^{k}(\deg P_i + 1) = r$. In what follows, a discrete function (of one variable) signifies a real-valued function with the variable taking on only nonnegative integral values. Define the difference operator Δ by

$$\Delta^0 y(x) = y(x),\ \Delta y(x) = y(x+1) - y(x),\ \Delta^n y(x) = \Delta(\Delta^{n-1}y(x)) \qquad (n = 2,3,\ldots).$$

It readily follows that

$$\Delta^n y(x) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} y(x+i) \qquad (n = 1,2,\ldots).$$

Given n discrete functions $f_1(x),\ldots,f_n(x)$, the Casorati's determinant of $f_1 ,\ldots, f_n$ is defined to be (see p. 354 of Milne-Thomson [3])

$$D(f_1 ,\ldots, f_n) := \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1(x+1) & f_2(x+1) & \ldots & f_n(x+1) \\ \ldots & & & \\ f_1(x+n-1) & f_2(x+n-1) & \ldots & f_n(x+n-1) \end{vmatrix}$$

By adding and subtracting appropriate rows, we see that also

$$D(f_1 ,\ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ \Delta f_1(x) & \Delta f_2(x) & \ldots & \Delta f_n(x) \\ \ldots & & & \\ \Delta^{n-1} f_1(x) & \Delta^{n-1} f_2 & \ldots & \Delta^{n-1} f_n(x) \end{vmatrix}$$

The determinant

$$D_{r+1}(f) = D_{r+1}(f(x)) := \begin{vmatrix} f(x) & \Delta f(x) & \ldots & \Delta^r f(x) \\ \Delta f(x) & \Delta^2 f(x) & \ldots & \Delta^{r+1} f(x) \\ \ldots & & & \\ \Delta^r f(x) & \Delta^{r+1} f(x) & \ldots & \Delta^{2r} f(x) \end{vmatrix}$$

will play the same role as $H_{r+1}(f)$ does in the differential case.

Theorem 1. Let f be a discrete function. For all nonnegative integral x, suppose that $D_1(f),\dots,D_r(f)$ are never zero. Then f satisfies $D_{r+1}(f) \equiv 0$ if and only if f is a discrete exponential polynomial of order r.

Proof. If f is a discrete exponential polynomial of order r, then (see Chapter XIII of Milne-Thomson [3]) f satisfies an exact r^{th} order linear difference equation with constant coefficients of the form

$$0 = \prod_{i=1}^{k}(\Delta-\alpha_i)^{r_i} y = (\Delta^r + b_1\Delta^{r-1} + \dots + b_r)y ,$$

where α_i are distinct real numbers, b_i real numbers, r_i positive integers, and $\sum_{i=1}^{k} r_i = r$. Taking differences of this equation r times successively, we obtain a system of r+1 linear homogeneous equations for the quantities 1, $b_1,\dots, b_r$. The determinant of this system is $D_{r+1}(f)$, which must then vanish.

To establish the other implication, we first look at the special case $r = 1$. Suppose that

$$D_2(f) = \begin{vmatrix} f(x) & \Delta f(x) \\ \Delta f(x) & \Delta^2 f(x) \end{vmatrix} \equiv 0 .$$

Then

$$f(x)\Delta^2 f(x) - (\Delta f(x))^2 = 0 \qquad (2)$$

Since $D_1(f) = f(x) \neq 0$ for all nonnegative integral x, then (2) is equivalent to (see problem 6, p. 50 of Milne-Thomson [3])

$$\Delta\left(\frac{\Delta f(x)}{f(x)}\right) = \frac{f(x)\Delta^2 f(x) - (\Delta f(x))^2}{f(x)f(x+1)} = 0 .$$

Then for some real constant C_1, we have $Df(x) = C_1 f(x)$. Solving this last difference equation, we get $f(x) = C_2(1 + C_1)^x$, for some nonzero real constant C_2. The assertion in this case then follows.

To prove the general case, let $c_0(x),c_1(x),\dots,c_r(x)$ be the cofactors of the last row of $D_{r+1}(f)$, then by the property of cofactors,

$$\sum_{i=0}^{r} c_i(x)\Delta^{i+m} f(x) = 0 \qquad (m = 0,1,\dots,r-1),$$

and

$$c_0(x)\, D^r f(x) + \dots + c_r(x)\, D^{2r} f(x) = D_{r+1}(f) \equiv 0 .$$

From the hypotheses, the coefficients $c_r(x) = D_r(f) \neq 0$ for all nonnegative integral x, so we divide by $c_r(x)$ and get for $m = 0,1,\dots,r-1$

$$\sum_{i=0}^{r} d_i(x)\Delta^{i+m} f(x) = 0 \qquad (3)$$

where $d_i(x) = c_i(x)/c_r(x)$ $(i = 0,1,\dots,r-1)$, $d_r(x) = 1$. Taking differences of these equations corresponding to $m = 0,1,\dots,r-1$, we get

$$\sum_{i=0}^{r}\left\{(\Delta d_i(x))\Delta^{i+m} f(x+1) + d_i(x)\Delta^{i+m+1} f(x)\right\} = 0 \qquad (m = 0,1,\dots,r-1).$$

Using (3) and $\Delta d_r(x) = 0$, we see that

$$\sum_{i=0}^{r-1}(\Delta d_i(x))\Delta^{i+m}f(x+1) = 0 \qquad (m = 0,1,\ldots,r\text{-}1).$$

Considering these as equations for determining $\Delta d_0(x),\ldots,\Delta d_{r-1}(x)$, we see that the coefficient matrix is $D_r(f(x+1))$ which never vanishes by the hypotheses. Thus for all nonnegative integral x, $\Delta d_0(x) = \ldots = \Delta d_{r-1}(x) = 0$ implying that all $d_i(x)$ $(i = 0,1,\ldots,r\text{-}1)$ are constants. Now the case m = 0 of (3) and the fact that $d_r(x) = 1$ yield that f(x) satisfies a linear difference equation of exact order r, and hence f(x) must be a discrete exponential polynomial of order r.

Remarks

1. In contrast to the differential case [2], note the extra hypotheses that $D_1(f),\ldots,D_r(f)$ are nonzero for all nonnegative integral x. The differential case can do without this extra condition by invoking upon continuity. That this restriction is essential in the difference situation is illustrated by the folowing example.

Define $f(0) = f(1) = 1$, $f(x) = 0$ for all positive integral $x \geq 2$.

Evidently, f is not a discrete exponential polynomial and

$D_1(f(x))$	$=$	1	if $x = 0, 1$
	$=$	0	otherwise
$D_2(f(x))$	$=$	-1	if $x = 0$
	$=$	0	otherwise
$D_{r+1}(f(x))$	$\equiv$	0	for all integral $r \geq 2$, $x \geq 0$.

2. Unlike Wronskians, Casorati's determinats do not enjoy the beautiful identity derivable from Sylvester's determinant identity as alluded to in the proof of Theorem 1 of Karlin and Loewner [2].

3. The second part of the above proof resembles not only that of Karlin and Loewner but also the proof of Casorati's theorem on pp. 354-355 in Milne-Thomson [3].

THE SECOND MAIN THEOREM

Theorem 2. Let f be a discrete function. Suppose $D_1(f),\ldots,D_r(f)$ do not vanish for all nonnegative integral x. If f satisfies $D_{r+1}(f) \equiv 0$, while at $x = x_0$ (x_0 being a nonnegative integer) we have

$$D_1(f) > 0,\ D_2(f) > 0,\ \ldots,\ D_r(f) > 0 \qquad (4)$$

then $f(x) = \sum_{i=1}^{r} a_i\beta_i^x$, where $a_i > 0$, and b_i are real and distinct $(i = 1,\ldots,r)$.

Proof. By Theorem 1, f is a discrete exponential polynomial of order r, say $f(x) = \sum_{i=1}^{k} P_i(x)\alpha_i^x$, where α_i are real and distinct, P_i real polynomials $(i = 1,\ldots,k)$ and $\sum_{i=1}^{k}(\deg P_i + 1) = r$. Without loss of generality, we may assume $x_0 = 0$, for otherwise consider $f(x\text{-}x_0)$ instead of f(x). As in

Section 2, there exist positive constants b_i and distinct real β_i $(i = 1,\ldots,r)$ such that the discrete exponential polynomial $g(x) = \sum_{i=1}^{r} b_i\beta_i^x$ satisfies

$$\Delta^j f(0) = \Delta^j g(0) = \sum_{i=1}^{r} b_i(\beta_i - 1)^j \qquad (j = 0,1,\ldots,2r-1).$$

This immediately implies that $f(x) = g(x)$ $(x = 0,1,\ldots,2r-1)$.

The exponential polynomial $f(x) - g(x)$ then vanishes at 2r consecutive integral points, and by a result on generalized Vandermonde determinants corresponding to taking differences (see p. 283 of van der Poorten [4]), we know that $f(x) - g(x)$ must vanish identically, and the theorem is thus proved.

Remarks

A difference analogue of III mentioned in the introduction is relatively trivial as the existence of $\Delta^r f(0)$ for all nonnegative integral r trivially induces the existence of $D^r f(x)$ for all nonnegative integers r and x.

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บทคัดย่อ

ให้ f เป็นฟังก์ชันค่าจริงที่เกลามากเพียงพอ คาร์ลินและโลว์เนอร์ได้พิสูจน์ผลน่าสนใจสองสิ่ง ซึ่งกล่าวโดยย่อได้ว่า

(1) f เป็นพหุนามชี้กำลัง เมื่อและต่อเมื่อ วรอนสเกียนที่แน่นอนบางตัวเป็นศูนย์

(2) f เป็นผลบวกชี้กำลังที่มีสัมประสิทธิ์เป็นบวก เมื่อวรอนสเกียนที่มีอันดับแน่นอนอันดับหนึ่งเป็นศูนย์ ในขณะที่วรอนสเกียนอันดับต่ำกว่าเป็นบวก ณ จุด ๆ หนึ่ง

งานวิจัยนี้แสดงการพิสูจน์ (2) ด้วยวิธีที่แตกต่างจากเดิมไปเล็กน้อย และแสดงการค้นหาผลประเภทเดียวกันกับผลทั้งสองในโครงสร้างเต็มหน่วย โดยใช้ตัวกำหนดของคาโซราทีแทนที่วรอนสเกียน