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ON THE RELIABILITY OF A REPAIRABLE PARALLEL SYSTEM

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ABSTRACT

We analyse the reliability of Gaver's parallel system sustained by a cold standby unit and attended by two identical repairmen. The system satisfies the usual conditions (i.i.d. random variables, perfect repair, instantaneous and perfect switch, queueing). Each operative unit has a constant failure rate and an arbitrary repair time distribution.

Our analysis is based on a time dependent version of the supplementary variable method.

The basic partial differential equations are transformed into an integro-differential equation of the (mixed) Fredholm type.

The equation generalizes the integro-differential equation of Takàcs.

A particular case motivates the proposed analysis.

INTRODUCTION

Two-unit parallel systems (for instance two power generators, in active redundancy[1], connected with the light-plant of a tunnel) are widely used to increase the reliability of industrial plants. Gaver's two-unit parallel system[2] sustained by a cold or warm standby unit and attended by a **single** repair facility, henceforth called an S-system, has received considerable attention[3-9].

As a variant, we analyse the reliability of Gaver's parallel system sustained by a cold standby unit and attended by **two** identical repairmen, henceforth called a T-system. The T-system satisfies the usual conditions (i.i.d. random variables, perfect repair[10], instantaneous and perfect switch[1], queueing).

Each operative unit has a constant failure rate and a general repair time distribution. Both repairmen are jointly busy if and only if at least two units are in failed state. In any other case, at least one repairman is idle.

It is evident that the T-system reduces the waiting time for repair with respect to a similar S-system. Therefore, a T-system improves the reliability of the corresponding S-system.

Our reliability analysis is based on a time dependent version of the supplementary variable method. The partial differential equations are transformed into an integro-differential equation of the (mixed) Fredholm type. The equation generalizes Takàcs' integro-differential equation, e.g.[11].

A particular example motivates the proposed analysis.

FORMULATION

Consider a T-system satisfying the usual conditions.

Each operative unit has a constant failure rate $\lambda>0$ and a general repair time distribution $R(t)$, $R(o)=0$. Let $R_c(t)\equiv1-R(t)$. Without loss of generality, (see forthcoming remark) we may assume that R(t) has a density function defined on $[0, \infty)$.

Both repair facilities are jointly busy if and only if at least two units are in failed state. In any other case, at least one repairman is idle.

Let $\{N_t, t \geq 0\}$ be a stochastic process with arbitrary state space $\{A, B, C, D\} \subset \Re$ characterized by the following events :

$\{N_t = A\}$: "Both repairmen are idle at time t, i.e., two units are operating in parallel sustained by a cold standby unit."

$\{N_t = B\}$: "One repairman is busy at time t, i.e., two units are operating in parallel and one unit is in repair."

$\{N_t = C\}$: "Both repairmen are jointly busy and only one unit is operative at time t."

$\{N_t = D\}$: "Both repairmen are jointly busy and a failed unit is waiting for repair at time t."

Define the stopping time

$$\theta := \inf \{t > 0 : N_t = D | N_o = A\}.$$

In reliability engineering, θ is usually called the first system-down time.

It is plain that the behaviour of the process $\{N_t, t \geq 0\}$ **after** θ is irrelevant for system's reliability analysis. Therefore, it is obvious to consider the system-down state D as an absorbing state of $\{N_t\}$.

Furthermore, let $\{X_t, Y_t\}$ be a random permutation of the past repair times of failed units in progress at time t.

The process $\{N_t, X_t, Y_t, t \geq 0\}$ is a piecewise-linear Markov process with state space:

$\{N_t\}$, $N_t = A$ (the renewal state), $N_o = A$ a.s.,

$\{N_t, X_t\}$, $N_t = B$, $X_t \geq 0$,

$\{N_t, X_t, Y_t\}$, $N_t = C$, $X_t \geq 0$, $Y_t \geq 0$,

and absorbing state D.

For K=A, B, C, D define $p_K(t) := P\{N_t=K, \forall u: 0<u<t, N_u \neq D\}$, where $\forall t\geq0$, $p_A(t)+p_B(t)+p_C(t)+p_D(t)=1$.

Finally, let

$p_B(t,x)dx := P\{N_t = B, \forall u: 0<u<t, N_u \neq D, x<X_t \leq x+dx\}$,

$p_C(t,x,y)dxdy := P\{N_t = C, \forall u: 0<u<t, N_u \neq D, x<X_t \leq x+dx, y<Y_t\leq y+dy\}$.

Note that $p_D(t)=P\{\theta \leq t\}$. Hence, the reliability or survival function of the system is given by $P\{\theta > t\} = 1-p_D(t)$.

INTEGRO-DIFFERENTIAL EQUATION

In order to construct a set of partial differential equations, we apply the usual technical manipulations related to the supplementary variable method[11]. For $t>0$, respectively $t>x>0$, $t>y>0$, we obtain the so-called Kolmogorov equations.

$$\left(2\lambda + \frac{d}{dt}\right) p_A(t) = \int_0 p_B(t,x)\,\frac{dR(x)}{R_c(x)}\,, \tag{1}$$

$$\left(2\lambda + \frac{dR}{dx}\frac{1}{R_c(x)} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) p_B(t,x) = 2\,!\int_0^t p_C(t,x,y)\,\frac{dR(y)}{R_c(y)}\,, \tag{2}$$

$$\left(\lambda + \frac{dR}{dx}\frac{1}{R_c(x)} + \frac{dR}{dy}\frac{1}{R_c(y)} + \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) p_C(t,x,y) = 0\,, \tag{3}$$

$$p_D(t) = \lambda \int_0^t\int_0^z\int_0^z p_C(z,x,y)\,dx\,dy\,dz. \tag{4}$$

Remark

Clearly, $dp_D(t) = \lambda p_C(t)dt$, $\mathcal{L}$ - a.e.

Hence, the survival function $P\{\theta > t\}$ is absolutely continuous (with respect to the Lebesgue measure) **irrespective** of the canonical structure of R. Therefore, in order to keep the analysis as simple as possible, we have assumed, without loss of generality, the existence of a repair time density on $[0, \infty]$.

The boundary conditions are

$$\begin{aligned} p_A(0) &= 1, \\ p_B(t,0) &= 2\lambda p_A(t), \\ 2!\,p_C(t,x,0) &= \begin{cases} 2\lambda p_B(t,x), & \text{if } t \geq x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{5}$$

Using the method of characteristics[12], or a conditional probabilistic argument[13], yields by equation (3),

$$p_C(t,x,y) = \begin{cases} p_C(t-y,\ x-y,\ 0)e^{-\lambda y}\dfrac{R_c(x)R_c(y)}{R_c(x-y)}, & \text{if } t \geq x \geq y \geq 0, \\ p_C(t-x,\ 0,\ y-x)e^{-\lambda x}\dfrac{R_c(x)R_c(y)}{R_c(y-x)}, & \text{if } t \geq y \geq x \geq 0, \\ 0, \text{ otherwise.} \end{cases} \tag{6}$$

In order to simplify the equations, let

$$\Phi(u,v) := \begin{cases} p_C(u,\ v,\ 0)\dfrac{1}{R_c(v)}, & \text{if } u \geq v \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

But note that $p_C(u, v, 0) = p_C(u, 0, v)$.

By equations (1), (2), (5), (6) and some technical manipulations, we have

$$(2\lambda + \frac{d}{dt})p_A(t) = \lambda^{-1}\int_0^t \Phi t, y)dR(y),$$

$$p_B(t) = \lambda^{-1}\int_0^t \Phi(t, y)R_c(y)dy, \tag{7}$$

$$(2\lambda + \frac{\partial}{\partial t} + \frac{\partial}{\partial x})\Phi(t, x) = 2\lambda\int_0^x \Phi(t-y, x-y)e^{-\lambda y}dR(y) + 2\lambda e^{-\lambda x}\int_{y=x}^t \Phi(t-x, y-x)dR(y).$$

Observe that $\Phi(t, 0) = 2\lambda^2 p_A(t)$, $t \geq 0$.

Laplace transforms of functions, with respect to t, are denoted by the corresponding character marked with an asterisk.

For instance,

$$\Phi^*(s, u) := \int_{t=0}^{\infty} \Phi(t, u)e^{-st}dt, s > 0, u \geq 0.$$

By equations (7), Fubini's theorem and the obvious substitution $\Phi^*(s, u) = 2\lambda 2 p^*_A(s)\Psi^*(s, u)$, $u \geq 0$, we obtain the following

RESULT

For $s > 0$,

$$p_A^*(s) = \frac{1}{s+2\lambda\left(1 - \int_0^\infty \Psi^*(s, z)dR(z)\right)},$$

$$p_B^*(s) = \frac{2\lambda \int_0^\infty \Psi^*(s, z)R_c(z)\,dz}{s+2\lambda\left(1 - \int_0^\infty \Psi^*(s, z)dR(z)\right)},$$

where $\Psi^*(s, x)$, $x>0$ satisfies the integro-differential equation

$$\left(s+2\lambda+\frac{d}{dt}\right)\Psi^*(s, x) = 2\lambda \int_0^x \Psi^*(s, x-y)e^{-(s+\lambda)y}dR(y)+2\lambda e^{-(s+\lambda)x}\int_{y=x}^\infty \Psi^*(s, y-x)dR(y), \tag{8}$$

with the boundary condition $\Psi^*(s, 0)=1$.

Remark

By equation (4) we have $\mathbf{E}e^{-s\theta}=\lambda p_C^*(s)$, $s>0$.
However, the identity $p_A(t)+p_B(t)+p_C(t)+p_D(t)=1$, reveals that

$$Ee^{-s\theta} = \frac{\lambda}{\lambda+s}(1-sp_A^*(s)-sp_B^*(s)).$$

SOLUTION PROCEDURE

It is of interest to remark that the exact unique solution of equation (8) can be obtained by an application of the theory of sectionally holomorphic functions[13]. Unfortunately, the solution of the resulting Cauchy integral equation is, in general, extremely formal (Cf[13], pp 496-497).

In order to present computational results, we restrict here the solution procedure to the following important particular case.

Let

$$R(t) = \sum_{k=1}^{n} p_k (1-e^{-\lambda_k t}), n \geq 1, \lambda_k > 0, \sum_{k=1}^{n} p_k = 1 ,$$

where, without loss of generality, $\lambda_1 < .... < \lambda_n$.

We do **not** require that all p_k are positive (which is the case in the family of hyper-exponentials). As R(t) is supposed to be a probability distribution of a positive random variable, we must have

$$p_1 > 0 ; \sum_{k=1}^{n} p_k \lambda_k = 1 .$$

Note that, for instance,

$$(R_c(t))^{-1}; \quad n=2,\ p_1 > 0,\ p_2 < 0,\ p_1+p_2=1,\ \lambda_1 p_1+\lambda_2 p_2 \geq 0,$$

is log-convex, so that R(t) has an increasing repair rate[14].

Finally, let

$$\varphi(s, \omega) := \int_0^\infty \Psi^*(s, x) e^{-\omega x} dx, \omega \geq 0 .$$

Laplace transformation of equation (8) and inserting $\omega=\lambda j$ in the resulting equation, yields for j=1,....,n:

$$\left[s + 2\lambda(1-\sum_{k=1}^{n} p_k \frac{\lambda_k}{s+\lambda+\lambda+\lambda_j}) + \lambda_j \right] \varphi(s, \lambda_j) = 1 + 2\lambda \sum_{k=1}^{n} \varphi(s, \lambda_k)\, p_k \frac{\lambda_k}{s+\lambda+\lambda_k+\lambda_j} .$$

We have reduced the solution of the Fredholm equation to the solution of n linear equations in n unknowns $\{\varphi(s, \lambda j);\ j=1,....,n\}$.

Clearly,

$$Ee^{-s\theta} = \frac{\lambda}{\lambda+s}\left(1-s\frac{1+2\lambda \sum_{k=1}^{n} \varphi(s, \lambda_k) p_k}{s+2\lambda[1-\sum_{k=1}^{n} \varphi(s, \lambda_k) p_k \lambda_k]}\right) .$$

It is not hard to see that $\mathbf{E}e^{-s\theta}$ is a rational transform. Hence, the inverse easily follows from an appropriate computer routine.

NUMERICAL EXAMPLE

Let, for instance, $n=2$; $\lambda=0.1$; $\lambda_1=2$; $\lambda_2=20$; $p_1=1.1$; $p_2=-0.1$ Inverting the Laplace transform

$$\frac{1 - Ee^{-s\theta}}{s}, \text{Re } s > 0,$$

yields the reliability function

$$P\{\theta > t\} = 1.00037e^{-0.000518651t} - 4.69399\ 10^{-4}e^{-2.03573t} + 9.69742\ 10^{-5}\ e^{-4.50393t} + 4.84331\ 10^{-7}\ e^{-20.0075t} - 4.38906\ 10^{-7}\ e^{-22.2926t} + 1.29757\ 10^{-9}\ e^{-40.0597t}.$$

REFERENCES

1. Birolini, A.(1994). Quality and Reliability of Technical Systems. Springer-Verlag, Berlin.
2. Gaver, D.P. (1963). Time to failure and availability of paralleled systems with repair. *IEEE Trans. Reliab.* **R-12**, 30-38.
3. Dhillon, B.S. and Yang, N. (1993). Human error analysis of a redundant system with arbitrarily distributed repair times. *Microelectron. Reliab.* **33**, 431-444.
4. Dhillon, B.S. and Anude, O.C. (1993). Common-cause failure analysis of a parallel system with warm standby. Microelectron. *Reliab.* **33**, 1321-1342.
5. Dhillon, B.S. and Yang, N. (1993). Availability of a man-machine system with critical and non-critical human errors. Microelectron. *Reliab.* **33**, 1511-1521.
6. Dhillon, B.S. and Anude, O.C. (1994). Income optimization of a repairable and redundant system. *Microelectron. Reliab.* **34** , 1709-1720.
7. Dhillon, B.S. and Anude, O.C. (1994). Optimizing the income of a repairable reliability system with early standby activation. *Microelectron. Reliab.* **34**, 1721-1732.
8. Goel, L.R., Gupta, R. and Singh, S. (1984). A two (multi-component) unit parallel system with standby and common-cause failures. *Microelectron. Reliab.* **24**, 415-419.
9. Mahmound, M., Mokhles, N.A. and Saleh, E.H. (1989). A non-identical units redundant system with common-cause failure and one standby unit. *IAPQR Trans.* **14**, 15-28.
10. Gertsbakh I.B. (1989). Statistical Reliability Theory. Marcel Dekker, New York.
11. Cohen, J.W. (1982). The Single Server Queue. North-Holland, Amsterdam.
12. Ohashi, M. and Nishida, T. (1980). A two-unit paralleled system with general distributions. *J. Opns Res. Soc. Japan* **23**, 313-325.
13. Vanderperre, E.J. (1990). Reliability analysis of a two-unit parallel system with dissimilar units and general distributions. *Microelectron. Reliab.* **30**, 491-501.
14. Shaked, M. and Shanthikumar, J.G. (1990). Reliability and Maintainability. In: Handbook in Operations Research and Management Science 2. (Heyman, D.P. and Sobel, M.J. Eds.) North-Holland, Amsterdam.