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ITERATIVE PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper two theorems of iterative partial differential equation have been proved. One of them is theorem of uniqueness and the other is theorem of existence. An example is given.

I. GENERAL FORMULA

The iterative partial differential equation, on the set Z, is of the form

$$(1) \qquad \frac{\partial^n u(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = f(x,u(x),u^2(x),\ldots,u^m(x))$$

with

$$(2) \qquad u(x) = g(x) \text{ on the boundary of } Z,$$

where m is an integer greater than 1 and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}, \quad u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ . \\ u_n(x) \end{bmatrix} \text{ and}$$

$$f(x, u(x), u^2(x),\ldots,u^m(x)) = \begin{bmatrix} f_1(x,u(x),u^2(x),\ldots,u^m(x)) \\ f_2(x,u(x),u^2(x),\ldots,u^m(x)) \\ . \\ f_n(x,u(x),u^2(x),\ldots,u^m(x)) \end{bmatrix}$$

$$\text{and } u^2(x) = u(u(x)) = \begin{bmatrix} u_1(u_1(x),u_2(x),\ldots,u_n(x)) \\ u_2(u,(x),u_2(x),\ldots,u_n(x)) \\ . \\ u_n(u,(x),u_2(x),\ldots,u_n(x)) \end{bmatrix}$$

$$u^3(x) = u(u^2(x))$$

.

$$u^m(x) = u(u^{m-1}(x))$$

$$Z = [o,a_1] \times [o,a_2] \times \ldots \times [o,a_n] \text{ , } a_1, a_2, \ldots, a_n \text{ are in } R^+$$

$$u_i : Z \to R \text{ , } f_i : D \to R \text{ , } i = 1,2,\ldots,n$$

$$u : Z \to R^n \text{ , } f : D \to R^n$$

D is a suitable subset of $Z \times R^{mn}$.

If f and g are continuous in D then the problem(1)-(2)is equivalent to the problem of the existence of the continuous solution of the integral equation

$$(3) \qquad u(x) = g(x) + \int_o^x f(t, u(t), u^2(t),\ldots,u^m(t))dt.$$

II. UNIQUENESS

Let $f(x,z_1,z_2,\ldots,z_m)$ be defined and continuous in D and g be defined and continuous in Z and let

$$(4) \qquad ||f(x,z_1,z_2,\ldots,z_m)|| \leq K$$

$$(5) \qquad ||f(x,z_1,z_2,\ldots,z_m) - f(x, \bar{z}_1, \bar{z}_2,\ldots,\bar{z}_m)||$$

$$\leq M_1 ||z_1 - z_1|| + M_2 ||z_2 - z_2|| + \ldots + M_m ||z_m - \bar{z}_m||$$

$$(6) \qquad ||g(x)|| \leq L \leq K$$

for every $(x,z_1,z_2,\ldots,z_m)$, $(\bar{x},\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_m)$ in D and every x in Z and $K,L,M_1,M_2,\ldots,M_m$ in R^+. The norm$||.||$ is the Euclidean norm.

We are looking for the solution u(x) of the problem (1)-(2) or (4) where

(7) u(x) belongs to Z for all x in Z and

(8) $||u(x) - u(y)|| \leq N ||x-y||$ for all x,y in Z and N in R^+.

Let

$$S_1 = M_1 + M_2N + M_3N^2 + \ldots + M_m N^{m-1}$$

$$S_2 = M_2 + M_3N + M_3N + \ldots + M_m N^{m-2}$$

.

$$S_{m-1} = M_{m-1} + M_mN$$

$$S_m = M_m$$

and $a = a_1 a_2 \ldots a_n$,$b = S_2 + S_m + \ldots + S_m$, $d = S_1^{(1/m)}$,$d > 0$, $B = ab$, $C = a_1 + a_2 + \ldots + a_n$, B, C in R^+. Thus we have the following theorem.

Theorem 1. If $Be^{dC} < 1$ and f and g satisfy the above conditions then there exists at most one solution to the problem (1)-(2).

Proof. Suppose u and v are two solutions of the problem (1)-(2) and let

$$p(x) = ||u(x) - v(x)||$$

and

$$P = \max_{x \varepsilon Z} ||u(x)-v(x)||$$

Thus we have

$$p(x) = ||g(x) - \int_o^x f(t,u(t),u^2(t),...,u^m(t))dt - g(x) + \int_o^x f(t,v(t),v^2(t),...,v^m(t))dt||$$

$$\leq \int_o^x ||f(t,u(t),u^2(t),...,u^m(t)) - f(t,v(t),v^2(t),...,v^2(t))|| \, dt$$

$$\leq M_1 \int_o^x ||u(t) - v(t)|| \, dt + M_2 \int_o^x ||u^2(x)-v^2(t)|| \, dt +...+ M_m \int_o^x ||u^m(t)-v^m(t))|| \, dt.$$

But

$$||u^2(t) - v^2(t)|| \leq Np(t) + p(v(t))$$

$$||u^3(t) - v^3(t)|| \leq N^2p(t) + Np(v(t)) + p(v^2(t))$$

$$||u^m(t) - v^m(t)|| \leq N^{m-1}p(t) + N^{m-2}p(v(t))+...+Np(v^{m-2}(t)) + p(v^{m-1}(t))$$

thus we have

$$p(x) \leq \int^x [M_1 + M_2N + M_3N^2 +...+ M_m N^{m-1}]p(t)dt$$
$$+ \int^x [M_2 + M_3N + M_4N^2 +...+ M_mN^{m-1}]p(v(t))dt$$
$$.$$
$$+ \int_o^x [M_{m-1} + M_{mN}]p^{m-2}(v(t))dt$$
$$+ \int_o^x M \, p_m (v(t))dt$$

and then

$$p(x) < S_1 \int_o^x p(t)dt + S_2 \, {}_o\int^x p(v(t))dt +...+ S_m \int_o^x p(v^{m-1}(t))dt$$
$$\leq S_1 \int_o^x p(t)dt + a \, S_2 \, P + a \, S_3 \, P +...+ a \, S_m \, P$$
$$= S_1 \int_o^x p(t)dt + a \, P \, [S_2 + S_3 +...+ S_m]$$
$$= S_1 \int_o^x p(t)dt + ab \, P.$$

Then

$$p(x) \leq a \, b \, P \, e^{d(x_1 + x_2 +...+ a_n)}$$
$$\leq a \, b \, P \, e^{d(a_1 + a_2 +...+ a_n)}$$
$$\leq a \, b \, P \, e^{dC}$$
$$\leq B \, P \, e^{dC}$$

but by the hypothesis that $B\,e^{dC} < 1$ then $p(x) < P$ for all x in Z , thus p(x) must be equal to zero which ends the proof of theorem 1.

III. EXISTENCE

Let

$$W = a\,[M_1 + (N+1)\,M_2 + (N^2 + N + 1)M_3 + \dots + (N^{m-1} + N^{m-2} + \dots + N+1)M_m]$$

and let consider the sequence

$$(9) \qquad u_{k+1}(x) = g(x) + {}_o\int^{x} f(t, u_k(t), u^2(t),\dots,u_k(t))dt$$

where $u_o(x)$ is fixed function of class C^1 map Z to Z such that

$$\left\| \frac{\partial^n u_0(x)}{\partial x_1 \partial x_2 \dots \partial x_n} \right\| \le K.$$

Hence we have the following theorem.

Theorem 2. Let the conditions of theorem 1 hold and $W < 1$ then the sequence (9) converges uniformly to the (unique) solution $u = u(x)$ of the problem (1)-(2).

Proof. We put

$$U_k = \max_{x\,\varepsilon\,z} \, ||u_{k+1}(x) - u_k(x)||$$

then we have

$$U_k \le W^{k-1}\,U_o \,.$$

Thus U_k tends to zero as k tends to infinity. This means that if $\{u_a(.)\}$ is a subsequence of $\{u_k(.)\}$ tending uniformly to some u(.) then u(.) is a solution of the problem (1)-(2). Since the family$\{u_n\}$is an Arzela-Ascoli family, thus for every subsequence $\{u_k\}$ of $\{u_k\}$ there exists a subsequence $\{u_k\}$ uniformly convergent and the limit needs to be a solution of the problem (1)-(2) as it was mentioned above. Thus the sequence $\{u_n\}$ tend uniformly to the (unique) solution u of the problem (1)-(2). This ends the proof of theorem 2.

IV. EXAMPLE

Find the solution, in $Z = [0,1] \times [0,1]$ of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} -\frac{1}{4} - \frac{x}{8} - \frac{5y}{8} + \frac{5xy}{8} + u_1(x,y) - u_2(x,y) + u_1(u_1,u_2) + u_2(u_2,u_2) \\ \frac{1}{4} + \frac{x}{8} + \frac{5y}{8} - \frac{5xy}{8} - u_1(x,y) + u_2(x,y) - u_1(u_1,u_2) - u_2(u_2,u_2) \end{bmatrix}$$

with

$$g(x,y) = \left[\frac{x}{4} + \frac{y}{4} \, , \; \frac{x}{4} - \frac{y}{4} \right]$$

i.e. the initial conditions is that

$$u(x,0) = \begin{bmatrix} \frac{x}{4} \\ \\ \frac{x}{4} \end{bmatrix} , \; u(0,y) = \begin{bmatrix} \frac{y}{4} \\ \\ \frac{-y}{4} \end{bmatrix}$$

Let $u_o(x,y) = g(x,y)$ and by the equation (9) we get

$$u_1(x,y) = \begin{bmatrix} \frac{x}{4} + \frac{y}{4} - \frac{xy}{4} + \frac{5x^2y^2}{32} \\ \\ \frac{x}{4} - \frac{y}{4} + \frac{xy}{4} - \frac{5x^2y^2}{32} \end{bmatrix}$$

$$u_2(x,y) = \begin{bmatrix} \frac{x}{4} + \frac{y}{4} - \frac{xy}{4} + \frac{25x^3y^3}{576} \\ \\ \frac{x}{4} + \frac{y}{4} + \frac{xy}{4} - \frac{25x^2y^2}{576} \end{bmatrix}$$

$$u_k(x,y) = \begin{bmatrix} \frac{x}{4} + \frac{y}{4} - \frac{xy}{4} + \frac{5^k x^{k+1} y^{k+1}}{2^{k+1} \, 2^2 \, 3^2 \ldots (k+1)^2} \\ \\ \frac{x}{4} - \frac{y}{4} + \frac{xy}{4} - \frac{5^k x^{k+1} y^{k+1}}{2^{k+1} \, 2^2 \, 3^2 \ldots (k+1)^2} \end{bmatrix}$$

then $u_k(x,y)$ tends to $\left[\frac{x + y - xy}{4} \, , \; \frac{x - y + xy}{4} \right]^T$

as k tends to infinity. Thus the solution of the given equation is

$$\begin{bmatrix} \frac{x + y - xy}{4} \\ \\ \frac{x - y + xy}{4} \end{bmatrix}$$

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