J. Sci. Soc. Thailand, 15 (1989) 133-138

ALGEBRAIC INDEPENDENCE TEST OF ARITHMETIC FUNCTIONS USING JACOBIANS

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(Received 13 March 1989)

ABSTRACT

We have derived a test for algebraic independence (with respect to convolution) of arithmetic functions based on a criterion of Shapiro and Sparer which involves the use of Jacobians. This test is then applied to establish algebraic independence of (arithmetic) zeta and various d-free functions.

INTRODUCTION

An arithmetic function is a complex-valued function whose domain is the set of natural numbers IN. It is well known that the set A of all arithmetic functions forms a ring with respect to addition and convolution,[1-4] where the convolution of the two arithmetic functions f and g is defined by

$$(f*g)(n) = \sum_{ij=n} f(i)\, g(j)$$

A notion which has recently attracted more attention is that of *algebraic (in)dependence (over the field of complex number $\mathbb{C}$). A set of arithmetic functions $f_1,\dots,f_r$ is said to be *algebraically independent (over $\mathbb{C}$) if there exists no nontrivial polynomial P with complex coefficients such that

$$P(f_1,\dots,f_r) := \sum_{(i)} a_{(i)}\, f_1^{*i_1} *\dots* f_r^{*i_r} = 0$$

where $a_{(i)} \in \mathbb{C}, f^{*i} = f*f*\dots*f$ (i times).

Define the γth arithmetic zeta function by

$$\zeta_\gamma(n) = n^\gamma$$

and the γth square-free function by

$$Q_\gamma(n) = \begin{cases} n^\gamma & \text{, if n is square-free,} \\ 0 & \text{, otherwise.} \end{cases}$$

Carlitz,[5] Popken,[3,6] Shapiro and Sparer[7] showed that $\zeta_0,\ldots,\zeta_r,Q_0,\ldots,Q_s$ are *algebraically independent (over $\mathbb{C}$).In a recent note,[8] we have improved this result by showing that $\zeta_0,\ldots,\zeta_r$, $R_{d_1,0},\ldots,R_{d_1,s_1},\ldots,R_{d_m,0},\ldots,R_{d_m,s_m}$ are *algebraically independent (over $\mathbb{C}$),where $R_{d,\gamma}$ is the γth d-free function defined by

$$R_{d,\gamma}(n) = n^{\gamma} \quad ,\text{if } n \text{ is a d-free integer,}$$
$$= 0 \quad ,\text{otherwise;}$$

n being d-free means the highest power of any prime factor contained in n is d-1. In the proof there, we made use of an *algebraic dependence criterion of Popken,[6] and strategically reduced the problem to cases of fewer d-free functions. In the present paper, we give another proof of this result by a completely different method based on the use of Jacobians. We first derive a convenient modification of Shapiro and Sparer's *algebraic independence criterion,[7] and then apply it to prove the desired independence result.

MATERIALS AND METHODS

A derivation D over the ring of arithmetic functions A is a mapping of A into itself such that

$$D(f*g) = Df*g + f*Dg$$

and

$$D(c_1f + c_2g) = c_1Df + c_2Dg,$$

for all f,g $\in$ A, and complex constants c_1,c_2.[4] A typical example of derivation is the (p$-$) basic derivation, p prime, defined by

$$D_p(f)(n) = f(np)\, v_p(np),$$

where $v_p(n)$ denotes the exponent of the highest power of p which divides n. Given $f_1,\ldots,f_t$ in A and derivations $D_1,\ldots,D_t$ over A, the Jacobian of the f_i relative to the D_i is the tXt determinant

$$J(f_1,\ldots,f_t;D_1,\ldots,D_t) := \det(D_i(f_j)),$$

where each product in the determinant expansion is taken to be a convolution product.

In,[7] Shapiro and Sparer proved the following theorem, which is the starting point of our work.

Theorem. (Shapiro-Sparer). Let $f_1,\ldots,f_t$ be given functions of A and $D_1,\ldots,D_t$ derivations over A which annihilate all elements of a subring E of A. If $J(f_1,\ldots,f_t;D_1,\ldots,D_t) \neq 0$, then the $f_1,\ldots,f_t$ are *algebraically independent over E.

Let $p_1,\ldots,p_t$ be distinct primes and $D_1,\ldots,D_t$ their corresponding basic derivations. From Shapiro-Sparer's theorem above, if $D_1,\ldots,D_t$ annihilate all elements of a subring E of A, and if

$$J := J(f_1,\ldots,f_t;D_1,\ldots,D_t) = \begin{vmatrix} D_1f_1 & \cdots & D_1f_t \\ \vdots & & \vdots \\ D_tf_1 & \cdots & D_tf_t \end{vmatrix} \neq 0,$$

then $f_1,\ldots,f_t$ are *algebraically independent over E. Now $J \neq 0$ when and only when there exists a natural number n such that

$$J(n) = \sum_{(i)} e_{(i)} (D_1 f_{i_1} *\ldots* D_t f_{i_t})(n) \neq 0,$$

where the sum is taken over all possible permutations

$(i) = (i_1,\ldots,i_t)$ of $(1,2,\ldots,t)$,

and $e_{(i)} = 1$ if (i) is an even permutation, and $= 0$, otherwise. Expanding the convolution product, and using the defining property of basic derivations, we get

$$J(n) = \sum_{(i)} e_{(i)} \sum_{k_1\ldots k_t = n} D_1 f_{i_1}(k_1)\ldots D_t f_{i_1}(k_1)\ldots D_1 f_{i_t}(k_t)$$

$$= \sum_{k_1\ldots k_t = n} \sum_{(i)} e_{(i)} f_{i_1}(k_1p_1)\ldots f_{i_t}(k_tp_t) V_{p_1}(k_1p_1)\ldots v_{p_t}(k_tp_t)$$

$$= \sum_{k_1\ldots k_t = n} v_{p_1}(k_1p_1)\ldots v_{p_t}(k_tp_t) \begin{vmatrix} f_1(k_1p_1) & \cdots & f_1(k_tp_t) \\ \vdots & & \vdots \\ f_t(k_1p_1) & \cdots & f_t(k_tp_t) \end{vmatrix}$$

RESULTS

The result obtained at the end of the last section can be formulated as our first main theorem.

Theorem 1. Let $f_1,\ldots,f_t$ be given functions of A. Suppose that there exist distinct primes $p_1,\ldots,p_t$ whose basic derivations annihilate all elements of a subring E of A. If there exists a natural number n such that

$$\sum_{k_1\ldots k_t = n} v_{p_1}(k_1p_1)\ldots v_{p_t}(k_tp_t) \begin{vmatrix} f_1(k_1p_1) & \cdots & f_1(k_tp_t) \\ \vdots & & \vdots \\ f_t(k_1p_1) & \cdots & f_t(k_tp_t) \end{vmatrix} \neq 0,$$

then $f_1,\ldots,f_t$ are *algebraically independent over E.

Using this theorem, we now derive a test which is more convenient to apply. Take E to be the subring of A which is isomorphic to $\mathbb{C}$ i.e.

$\{ f \in A;\ f(n) = c \in \mathbb{C}$ if $n = 1$ and $f(n) = 0$ otherwise.$\}$.

Clearly, then, for each prime p, the corresponding basic derivation D_p annihilates all elements of $\mathbb{C}$ We thus have:

Corollary. Let $f_1,...,f_t$ be given functions of A. Let $p_1,...,p_t$ be distinct primes and $D_1,...,D_t$ their corresponding basic derivations. If there exists a natural number n such that

$$\sum_{k_1...k_t = n} v_{p_1}(k_1p_1)...v_{p_t}(k_tp_t) \begin{vmatrix} f_1(k_1p_1) & \cdots & f_1(k_tp_t) \\ \vdots & & \vdots \\ f_t(k_1p_1) & \cdots & f_t(k_tp_t) \end{vmatrix} \neq 0,$$

then $f_1,...,f_t$ are *algebraically independent over $\mathbb{C}$

We are now ready to establish our second main result.

Theorem 2. Let $m(\geqq 1)$, $d_1 > ... > d_m \geqq 2$, $s_0,s_1,...,s_m$ be nonnegative integers. Then the arithmetic functions $\zeta_0,...,\zeta_{s_0},R_{d_1,0},...,R_{d_1 s_1},...,R_{d_m,0},...,R_{d_m,s_m}$ are *algebraically independent over $\mathbb{C}$ Proof. Let

$$i_\alpha = (i_{\alpha 0}, i_{\alpha 1}, ..., i_{\alpha s_\alpha}) \qquad (\alpha = 0,1,...,m)$$

be m+1 vectors whose components are nonnegative integers. Let

$$(p_{\alpha\beta} : \alpha = 0,1,...,m;\ \beta = 0,1,...,s_\alpha)$$

be a sequence of $\sum_{\alpha=0}^{m} (s_\alpha + 1)$ distinct primes. Consider the function

$$f(i_0,i_1,...,i_m) = \det(A_{\alpha\beta}) \quad \begin{matrix} \alpha = 0,1,...,m \\ \beta = 0,1,...,m \end{matrix}$$

where the $A_{\alpha\beta}$'s are $(s_\alpha + 1) \times (s_\beta + 1)$ submatrices defined by

$$A_{0\beta} = \begin{bmatrix} \zeta_0(i_{\beta 0}p_{\beta 0}) & \cdots & \zeta_0(i_{\beta s_\beta} p_{\beta s_\beta}) \\ \vdots & & \vdots \\ \zeta_{s_0}(i_{\beta 0}p_{\beta 0}) & \cdots & \zeta_{s_0}(i_{\beta s_\beta} p_{\beta s_\beta}) \end{bmatrix} \quad (\beta = 0,1,...,m)$$

$$A_{\alpha\beta} = \begin{bmatrix} R_{d_\alpha,0}(i_{\beta 0}p_{\beta 0}) & \cdots & R_{d_\alpha,0}(i_{\beta s_\beta} p_{\beta s_\beta}) \\ \vdots & & \vdots \\ R_{d_\alpha,s_\alpha}(i_{\beta 0}p_{\beta 0}) & \cdots & R_{d_\alpha,s_\alpha}(i_{\beta s_\beta} p_{\beta s_\beta}) \end{bmatrix} \quad (\alpha = 1,2,...,m;\ \beta = 0,1,...,m).$$

Now consider the product

$$\prod_{\alpha=0}^{m} \prod_{\beta=0}^{s_\alpha} i_{\alpha\beta} = \prod_{\alpha=0}^{m-1} \prod_{\beta=0}^{s_\alpha} p_{\alpha\beta}^{d_{\alpha+1}-1} \qquad (1)$$

and recall that

$$\zeta_\alpha(n) = R_{d_{\alpha,\beta}}(n) \qquad (\alpha = 0,1,\ldots,m-m'; \beta = 0,1,\ldots,s_\alpha)$$

$R_{d_\alpha,\beta}(n) = 0$ $(\alpha = m-m'+1,\ldots,m;\ \beta = 0,1,\ldots,s_\alpha)$ if n is $d_{m-m'}$ – free, but not $d_{m-m'+1}$ – free (and so not $d_{m-m'+2},\ldots,d_m$ – free), where $m' = 0,1,\ldots,m$. Observe that among all possible $\sum_{\alpha=0}^{m}(s_\alpha + 1)$ – tuples $(i_0,\ldots,i_m)$ of integers for which the relation (1) holds, all but one of their corresponding determinant values F $(i_0,\ldots,i_m)$ vanish, because of two identical rows or two identical columns. The only surviving determinant has

$$i_{00} = p_{00}^{d_1-1},\ldots, i_{0s_0} = p_{0s_0}^{d_1-1}$$

$$i_{10} = p_{10}^{d_2-1},\ldots, i_{1s_1} = p_{1s_1}^{d_2-1}$$

$$\vdots$$

$$i_{m-1,0} = p_{m-1,0}^{d_m-1},\ldots, i_{m-1,s_{m-1}} = p_{m-1,s_{m-1}}^{d_m-1}$$

$$i_{mo} = i_{ml} = \ldots = i_{ms_m} = 1$$

with value

$$F := F(p_0(d_1), p_1(d_2),\ldots,p_{m-1}(d_m), 1)$$

$$= \begin{vmatrix} A_{00}(p_0(d_1)) & \cdot & \cdot & \cdot \\ 0 & A_{11}(p_1(d_2)) & \cdot & \cdot \\ \vdots & & \ddots & \\ 0 & 0 \quad \cdots & A_{m-1,m-1}(p_{m-1}(d_m)) \ldots & \\ 0 & 0 \quad \cdots & 0 & A_{mm}(1) \end{vmatrix}$$

where the $A_{\beta\beta}(p(d_{\beta+1}))$ are square submatrices obtained from $A_{\beta\beta}$ by substituting $i_{\beta 0}, i_{\beta 1},\ldots,i_{\beta s_\beta}$ with appropriate values of prime powers as above. Since the block determinant of F has an upper triangular shape, expanding via Laplace's expansion of block determinants,[9] we arrive at

$$F = \pm \det (A_{mm} (I)) \prod_{\beta = 0}^{m-1} \det (A_{\beta\beta} (p_\beta (d_{\beta+1}))).$$

Each subdeterminant on the right hand side is a Vandermonde determinant and so does not vanish. Hence, $F \neq 0$. Invoking upon the corollary, the theorem follows.

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บทคัดย่อ

ส่วนแรกของงานนี้เป็นการสร้างเกณฑ์ทดสอบความเป็นอิสระเชิงพีชคณิต (เทียบกับการประสาน) ของฟังก์ชันเลขคณิต ที่มีรากฐานอยู่บนวิธีการของ Shapiro และ Sparer ซึ่งอาศัยการใช้ Jacobian จากนั้นจึงประยุกต์ใช้เกณฑ์นี้ในการพิสูจน์ความเป็นอิสระเชิงพีชคณิตของฟังก์ชันเลขคณิต zeta กับฟังก์ชัน *d*-อิสระ อื่น ๆ