RESEARCH ARTICLES

REDUCED DENSITY MATRICES
AND THERMO-HYDRODYNAMIC EQUATIONS OF HE II*

KITT VISOOTTIVISETH

Department of Physics, Faculty of Science, Chulalongkorn University, Bangkok 10500, Thailand

Abstract

     The microscopic theory of He II has been developed, accounting for the Bose-Einstein condensation by the existence of "off-diagonal long-range order" (ODLRO) in the reduced density matrices. The derivation of a closed set of thermo-hydrodynamic equations, for the condensate and the depletion of He II in the bulk system, was thus attempted.

Introduction

     Einstein[1] was the first who referred to the phase transition of non-interacting quantum gas obeying Bose statistics as the Bose condensation. Later, London[2] proposed that the mechanism responsible for the Bose-Einstein condensation would also account for the lambda (quantum phase) transition and for the unusual dynamic properties (e.g., superfluidity) of superfluid helium (He II). This paper attempts to elaborate on London's idea that the thermo–hydrodynamic equations of He II can be derived from the microscopic theory, in which macroscopic dynamics is introduced and discussed in terms of reduced density matrices. The thermo-hydrodynamics of superfluid helium was found to be strikingly different from,

and more complex than, that of the condensed ideal Bose gas. This work on He II, of course, must reduce to the ideal Bose gas when the interaction is switched off.

Yang[3] has shown that there is an off-diagonal long-range order (ODLRO) of the reduced density matrices, first suggested by Penrose[4] in the coordinate space representation for the many-body system of interacting bosons. Frohlich[5] has been successful in obtaining the Navier-Stokes hydrodynamic equation as an exact result from the properties of reduced density matrices. These confirm that reduced density matrices must provide a capable tool in solving the superfluid helium problem. In the next section the definitions and some properties or reduced density matrices will be reviewed briefly, only for the purpose of this work, and some more hydrodynamic equations will be shown. More details on reduced density matrices have been presented in our previous works[6].

Reduced Density Matrices and Conservation Laws

Since a complete description of the quantum system in the mixed state is assumed to be contained in the density matrix[7],

$$D_N \ (\vec{x}_1', ..., \vec{x}_N', t; \vec{x}_1'', ..., \vec{x}_N'', t) = \sum_{i=1}^{\nu} w_i \ \psi_i \ (\vec{x}_1'', ..., \vec{x}_N', t) \ \psi_i \ (\vec{x}_1'', ..., \vec{x}_N'', t), \tag{1}$$

where w is the statistical "weights" or the quantum statistical probability that the system is characterized by state i, and ψ_i is the N-particle wave function which could be obtained exactly by solving the Schrodinger equation, $\hbar \frac{\partial}{\partial t} \psi_i = H \psi_i$. The operator H is a Hamiltonian operator of the N-body system of identical particle,

$$H = \frac{-\hbar^2}{2m} \sum_{l=1}^{N} \nabla_l^2 = \frac{1}{2} \sum_{l \neq m} V(\vec{x}_l - \vec{x}_m), \tag{2}$$

where the particles interact in pairs via the spherically symmetrical two-body potential V (r). The combined quantum and statistical expectation of an observable, such as the average density $< p \ (x) >$ of the system of N point particles, is

$$< \rho (\vec{x}) > = \sum_{i=1}^{\nu} w_i \int .. \int \psi_i^* (\vec{x}_1, \vec{x}_2, .., \vec{x}_N, t) \sum_{l=1}^{N} \delta (\vec{x} - \vec{x}_l) \ \psi_i \ (\vec{x}_1, \vec{x}_2, .., \vec{x}_N, t) \ d\vec{x}_1 ... d\vec{x}_N,$$

$$= N \sum_{i=1}^{\nu} w_i \int .. \int \psi_i^* (\vec{x}, \vec{x}_2, .., \vec{x}_N, t) \ \psi_i \ (\vec{x}, \vec{x}_2, .., \vec{x}_N, t) \ d\vec{x}_2 .. \ d\vec{x}_N, \tag{3}$$

where ν is the number of states in the system.

The development in time of the density matrix is contained in the Schrödinger equation and is the quantum Liouville equation or

$$i\hbar \frac{\partial}{\partial t} D_N = \{ \sum_{l=1}^{N} -\frac{\hbar^2}{2m} (\nabla_l'^2 - \nabla_l''^2) + \sum_{l < m = 2}^{N} [V(\vec{x}_l' - \vec{x}_m') - V (\vec{x}_l'' - \vec{x}_m'')] \} \ D_N. \tag{4}$$

Although quantum statistics is described by the density matrix, it is far too difficult to solve explicitly. The reduced density matrices, for equal times, are thus defined as follows :

$$\Omega_1 \ (\vec{x}_1 \ ; \ \vec{x}_1'') \ = \ \frac{N \ !}{(N-1) \ !} \int .. \int D_N \ (\vec{x}_1', \ \vec{x}_2, .., \vec{x}_N; \ \vec{x}_1'' \ \vec{x}_2, .., \vec{x}_N) \ d\vec{x}_2 .. d\vec{x}_N,$$

$$\Omega_2 \ (\vec{x}_1', \ \vec{x}_2'; \ \vec{x}_1'', \ \vec{x}_2'') \ = \ \frac{N \ !}{(N-2) \ !} \int .. \int D_N \ (\vec{x}_1', \ \vec{x}_2', \ \vec{x}_3, .., \vec{x}_N; \ \vec{x}_1'', \ \vec{x}_2'', \ \vec{x}_3, .., \vec{x}_N) \ d\vec{x}_3 .. d\vec{x}_N, \tag{5}$$

etc., for Ω_n the *nth* order reduced density matrix. Its coupled equations of motion follows from equation (4) by mathematical induction :

$$i\hbar \frac{\partial \Omega}{\partial t}_n = \ -\frac{\hbar^2}{2m} \sum_{r=1}^{n} \ (\nabla_r'^2 - \nabla_r''^2) \ \Omega_n + \frac{1}{2} \sum_{r,s} \ [V \ (\vec{x}_r' - \vec{x}_s') - V \ (\vec{x}_r'' - \vec{x}_s'')] \ \Omega_n$$

$$+ \int \sum_{r=1}^{n} \ [V \ (\vec{x}_r' - \vec{y}) - V \ (\vec{x}_r'' - \vec{y})] \ \Omega_{n+1} \ (\vec{x}_1', .., \vec{x}_n', \ \vec{y}; \ \vec{x}_1'', .., \vec{x}_n'', \vec{y}) \ d\vec{y}. \tag{6}$$

The equations of motion for Ω_1 and Ω_2 can thus be written :

$$i\hbar \frac{\partial \Omega}{\partial t}_1 \ (\vec{x}_1'; \ \vec{x}_1'') \ = \ -\frac{\hbar^2}{2m} \sum_{r=1}^{n} \ (\nabla_1'^2 - \nabla_1''^2) \ \Omega_1 \ (\vec{x}_1' \ ; \ \vec{x}_1'')$$

$$+ \int [V \ (\vec{x}_1' - y) - V \ (\vec{x}_1'' - \vec{y})] \ \Omega_2 \ (\vec{x}_1', \ \vec{y} \ ; \vec{x}_1'', \ \vec{y}) \ d\vec{y}, \tag{7}$$

and

$$i\hbar \frac{\partial \Omega}{\partial t}_2 \ (\vec{x}_1', \ \vec{x}_2'; \ \vec{x}_1'', \ \vec{x}_2'') \ = \ -\frac{\hbar^2}{2m} (\nabla_1'^2 + \nabla_2'^2 - \nabla_1''^2 - \nabla_2''^2) \ \Omega_2 \ + \ [V \ (\vec{x}_1' - \vec{x}_2') - V \ (\vec{x}_1'' - \vec{x}_2'')] \Omega_2$$

$$+ \int [V \ (\vec{x}_1' - \vec{y}) + V \ (\vec{x}_2' - \vec{y}) - V \ (\vec{x}_1'' - \vec{y}) - V \ (\vec{x}_2'' - \vec{y})]$$

$$\Omega_3 \ (\vec{x}_1', \ \vec{x}_2', \vec{y}; \ \vec{x}_1'', \ \vec{x}_2'', \vec{y}) \ d\vec{y}. \tag{8}$$

These last two equations are important for deriving the macroscopic laws of mass, momentum, and energy conservation.

From equations (1), (3), and (5), the mass density of fluid at point $\vec{x}$ is

$$\rho \ (\vec{x}) \ = \ \Omega_1 \ (\vec{x}; \ \vec{x}). \tag{9}$$

Use of equations (7) and (9) leads to the mass conservation law

$$\frac{\partial \rho}{\partial t} \ + \ \nabla \cdot \vec{j} \ = \ 0, \tag{10}$$

where the momentum density,

$$\vec{j} \ (\vec{x}) \ = \ -\frac{\hbar}{2mi} \lim_{\vec{x}_1' \to \vec{x}_1''} \ [(\nabla_1' - \nabla_1'') \ \Omega_1 \ (\vec{x}_1'; \ \vec{x}_1'')], \tag{11}$$

is equivalent to the expectation value of the operator,

$$\vec{j}(\vec{x}) = \frac{1}{2} \sum_{i=1}^{N} [\vec{p}_i \, \delta(\vec{x} - \vec{x}_i) + \delta(\vec{x} - \vec{x}_i) \, \vec{p}_i].$$

From the equation (7) of motion for Ω_1 and equation (11), the momentum conservation law is obtained

$$\frac{\partial \vec{j}}{\partial t} + \nabla \cdot \overleftrightarrow{\Pi} = 0, \tag{12}$$

where

$$\frac{d\vec{j}}{dt} = \frac{\hbar}{2mi} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1' - \nabla_1'') \frac{\partial}{\partial t} \, \Omega_1 \, (\vec{x}_1'; \vec{x}_1''),$$

and the stress tensor is defined via

$$\nabla \cdot \overleftrightarrow{\Pi} = -\frac{\hbar^2}{4m^2} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1' - \nabla_1'')(\nabla_1'^2 - \nabla_1''^2) \, \Omega_1 \, (\vec{x}_1'; \vec{x}_1'')$$

$$+ \frac{1}{2m} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1' - \nabla_1'') \int [V(\vec{x}_1' - \vec{y}) - V(\vec{x}_1'' - \vec{y})] \, \Omega_2 (\vec{x}_1', \vec{y}; \vec{x}_1'', \vec{y}) \, d\vec{y}. \tag{13}$$

From the Hamiltonian H of the N-body system, equation (2), the energy density ε, which is defined via $<H(\vec{x})> = \int \varepsilon \, d\vec{x}$, can be written as

$$\varepsilon(\vec{x}) = -\frac{\hbar^2}{2m} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1'^2 + \nabla_1''^2) \, \Omega_1 \, (\vec{x}_1'; \vec{x}_1'') + \int V(\vec{x} - \vec{y}) \, \Omega_2 (\vec{x}, \vec{y}; \vec{x}, \vec{y}) \, d\vec{y}. \tag{14}$$

Substituting equations (7) and (8) into the time derivative of equation (14) yields the energy conservation law

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \vec{Q} = 0, \tag{15}$$

where the energy flux density $\vec{Q}$ is defined via

$$\nabla \cdot \vec{Q} = -\frac{\hbar^2}{2m} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1'^2 + \nabla_1''^2) \frac{\partial \Omega_1}{\partial t} (\vec{x}_1'; \vec{x}_1'') + \int V(\vec{x} - \vec{y}) \frac{\partial \Omega_2}{\partial t} (\vec{x}, \vec{y}; \vec{x}, \vec{y}) \, d\vec{y},$$

$$= -\frac{\hbar^3}{2m^2 i} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1'^2 + \nabla_1''^2)(\nabla_1'^2 - \nabla_1''^2) \, \Omega_1 \, (\vec{x}_1'; \vec{x}_1'')$$

$$+ \frac{\hbar}{2mi} \lim_{\vec{x}_1' \to \vec{x}_1''} (\nabla_1'^2 + \nabla_1''^2) \int [V(\vec{x}_1' - \vec{y}) - V(\vec{x}_1'' - \vec{y})] \, \Omega_2(\vec{x}_1', \vec{y}; \vec{x}_1'', \vec{y}) \, d\vec{y}. \tag{16}$$

Fröhlich[5] has derived the Navier-Stokes equation as an exact result from the reduced density matrices, with neither continuum assumptions nor intermediate 'master equations' required,

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla P + \eta_2 \nabla^2 \vec{v} - \eta_1 \nabla \times \nabla \times \vec{v}, \tag{17}$$

where P is the pressure, and η_1, η_2 are the coefficients of viscosity. For a non-interacting system, a derivation of the Euler equation can be obtained from equation (7) immediately,

The conservation laws, equations (10), (12), and (15), are not yet closed. To obtain the closed set of these laws, the off-equilibrium behavior of Ω_1 and an expression giving Ω_2 as a function of Ω_1 must be known for substituting into equations (11), (13) and (16). However, for a non-interacting system, the expression for Ω_2 is not necessary.

Thermo-hydrodynamic Equations of He II

To obtain the thermo-hydrodynamic equations, we will discuss the reduced density matrices and ODLRO, and the macroscopic quantities for He II. The He II systems is considered as an interacting Bose system, but the concept of its "condensation" is more complicated than the ideal Bose gas. When the condensation has occurred, every atom contributes both to the condensate spreading throughout the volume occupied by the system in the presence of strong interaction between the [4]He atoms and to localized regions of higher density (localized to within about the average interatomic spacing, 4.5 A). Many theoretical estimates[8],[9] for the condensate density ρ_c of He II at 0 K are between 8% and 25% of the total density ρ, and these have been confirmed by many experimental works[10]. The condensate density ρ_c must therefore be distinguished from the superfluid density ρ_s. The superfluid density ρ_s is equal to the total density at 0 K. Both the condensate density and superfluid density decrease with increase of temperature from 0 K and vanish at the lambda transition temperature, T_λ. Therefore a two-fluid model was proposed[11],[12] for He II as the condensate and depletion model: the total density

$$\rho = \rho_c + \rho_d, \tag{18}$$

where ρ_d is called the depletion density, and ρ_d is equal to the total density ρ at T_λ. Penrose[4] hase pointed out how a condensate could be meaningfully defined in liquid [4] He by using the reduced density matrices. He has suggested that the first order reduced density matrix Ω_1 can be factorized for the Bose condensation. Yang[3] has called the existence of factorization. 'off-diagonal long-range order' (ODLRO) in the reduced density matrices. The appearance of the factorized part in Ω_1 for the condensation in HeII system is:

$$\Omega_1 (\vec{x}'; \vec{x}'') = \phi^* (\vec{x}'') \phi (\vec{x}') + \Lambda_1 (\vec{x}'; \vec{x}''), \tag{19}$$

where $\Lambda_1 (\vec{x}'; \vec{x}'') \to 0$ when $\vec{x}' - \vec{x}'' \to \infty$

Liquid ^{4}He as well as a "quantum liquid" is differentiated from ordinary liquid at below T_λ by the appearance of macroscopic wave function $\phi(\vec{x})$. The wave function $\phi(\vec{x})$ is the statistical average $<\psi(\vec{x})>$ over an ensemble such that it has a definite phase and amplitude[13]. $\psi(\vec{x})$ is the spinless boson annihilation operator for ^{4}He atoms. $|\phi(\vec{x})|^2$ is thus defined as the "condensate" density ρ_c, below T_λ, and $\phi(\vec{x})$ is referred as the "condensate (macroscopic) wave function":

$$\phi(\vec{x}) = [\rho_c(\vec{x})]^{\frac{1}{2}} \exp[i\Theta(\vec{x})], \tag{20}$$

where $\rho_c(\vec{x}) = |\phi(\vec{x})|^2$. The condensate velocity is thus defined via

$$\vec{v}_C = \frac{\hbar}{m}\nabla\Theta(\vec{x}). \tag{21}$$

The "depletion" density ρ_d is then given by

$$\rho_d(\vec{x}) = \Lambda_1(\vec{x};\vec{x}), \tag{22}$$

and Λ_1 is then written in terms of two real functions : an even function $\rho_d(\vec{x}';\vec{x}'')$ and an odd function $\chi(\vec{x}';\vec{x}'')$,

$$\Lambda_1(\vec{x}';\vec{x}'') = \rho_d(\vec{x}';\vec{x}'')\exp[i\chi(\vec{x}';\vec{x}'')], \tag{23}$$

where the phase χ vanishes when $\vec{x}' = \vec{x}''$. The depletion velocity is defined via the depletion current density

$$\rho_d\vec{v}_d = \frac{\hbar}{2mi}\lim_{\vec{x}'\to\vec{x}''}(\nabla'-\nabla'')\Lambda_1(\vec{x}';\vec{x}''). \tag{24}$$

The condensate current density is also defined as

$$\rho_c\vec{v}_c = \frac{\hbar}{2mi}\lim_{\vec{x}'\to\vec{x}''}(\nabla'-\nabla'')\phi*(\vec{x})\phi(\vec{x}). \tag{25}$$

Thus the total current density of the HeII system is

$$\vec{j} = \rho\vec{v} = \rho_c\vec{v}_c + \rho_d\vec{v}_d. \tag{26}$$

In the bulk system, the depletion 'bulk' density which is spatially independent is defined via

$$\Lambda_1(\vec{x}';\vec{x}'') = \Lambda_1(|\vec{x}'-\vec{x}''|) = \rho_d h(|\vec{x}'-\vec{x}''|)\exp[i\chi(|\vec{x}'-\vec{x}''|)], \tag{27}$$

where h(r) is real function and approaches zero at 0K when r is much greater than the average interatomic spacing.

Yang[3] has also shown that the factorization of $\Omega_1(\vec{x}';\vec{x}'')$ in the limit $|\vec{x}'-\vec{x}''| \to \infty$ implies the factorization of the second order reduced density matrix $\Omega_2(\vec{x}_1',\vec{x}_2';\vec{x}_1'',\vec{x}_2'')$ of the Bose system. A very plausible form[6,14] of Ω_2 which shows the presence of ODLRO may be written as

$$\Omega_2 \, (\vec{x}_1', \vec{x}_2'; \vec{x}_1'', \vec{x}_2'')$$

$$= \$_1 \, (|\vec{x}_1' - \vec{x}_2'|) \, \$_1 \, (|\vec{x}_1'' - \vec{x}_2''|) \, \phi * (\vec{x}_1'') \, \phi * (\vec{x}_2'') \, \phi \, (\vec{x}_1') \, \phi \, (\vec{x}_2')$$

$$+ \$_2(|\vec{x}_1' - \vec{x}_2'|)\$_2(|\vec{x}_1'' - \vec{x}_2''|) \, [\phi * (\vec{x}_1'') \, \phi \, (\vec{x}_2') \Lambda_1 \, (\vec{x}_2'; \vec{x}_2'') + \phi * (\vec{x}_1'')\phi(\vec{x}_2') \, \Lambda_1 \, (\vec{x}_1'; \vec{x}_2'')$$

$$+ \phi * (\vec{x}_2'') \, \phi \, (\vec{x}_1') \, \Lambda_1 \, (\vec{x}_2'; \vec{x}_1'') + \phi * (\vec{x}_2'') \, \phi(\vec{x}_2')\Lambda_1 \, (\vec{x}_1''; \vec{x}_1')] \; + \; \Lambda_2 \, (\vec{x}_1', \vec{x}_2'; \vec{x}_1'', \vec{x}_2''), \quad (28)$$

where $\$_1$ (r) and $\$_2$ (r) are the screening factors for the "core" condition. The function h (r) and the screening factors $\$$ (r) can be obtained from the work of McMillan[8]. The function Λ_2 is required to satisfy all conditions analogous to the conditions[6] of Ω_2.

Now we will present the thermo-hydrodynamic equations of motion for the bulk system of ^{4}He II. Use of the form of ϕ and Ω_2 from equation (20) and (28) in the equation (7), after taking the limit as $\vec{x}' - \vec{x}''| \to \infty$ and separating the sear plart and imaginary part along with the definition of macroscopic quantities, gives the equations of motion for $\vec{v}_c$, ρ_c and ρ_d:

$$\rho_d : \quad \frac{\partial \vec{v}_c}{\partial t} + (\vec{v}_c.\nabla)\vec{v}_c \; = \; -\nabla \left[\frac{-h^2}{8m^2} \frac{(\nabla\rho_c)^2}{\rho_c^2} + \frac{\hbar^2}{4m^2} \frac{\nabla^2\rho_c}{\rho_c} + \tilde{\mu}(\rho_c,\rho_d) - \alpha \frac{\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2, \right]$$

$$(29)$$

$$\frac{\partial \rho_c}{\partial t} + \nabla.(\rho_c\vec{v}_c) \; = \; -\frac{\alpha}{2\rho} \, \nabla.\left[\rho_c\rho_d(\vec{v}_d - \vec{v}_c) \right] \qquad (30)$$

Since equation (10) becomes

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho_c v_c + \rho_d v_d) \; = \; 0, \qquad (31)$$

we thus obtain, from equations (30) and (31)

$$\frac{\partial \rho_d}{\partial t} + \nabla.(\rho_d v_d) \; = \; \frac{\alpha}{2\rho} \, \nabla.\left[\rho_c\rho_d(v_d - v_c) \right], \qquad (32)$$

where $\tilde{\mu}(\rho_c,\rho_d) \; = \; \frac{\rho_c}{m}\int V(r)\$_1(r)d^3r \; + \; \frac{\rho_d}{m}\int V(r)\$_2(r)d^3r \; + \; \frac{\rho_d}{m}\int V(r)\$_2(r)h(r)d^3r, \quad (33)$

and $\qquad \alpha \; = \; \frac{m}{3\hbar^2} \rho \int V(r)\$_2(r)h(r) \, r^2d^3r. \qquad (34)$

Since the hydrodynamic equations of the two-fluid model must be based on the hydrodynamic equations of a single fluid, the Navier-Stokes equation (17) as derived microscopically by Frohlich can thus be written for HeII as

$$\rho_c\frac{\partial \vec{v}_c}{\partial t} + \rho_c(\vec{v}_c.\nabla)\vec{v}_c + \rho_d\frac{\partial \vec{v}_d}{\partial t} + \rho_d(\vec{v}_d.\nabla)\vec{v}_d \; = \; -\nabla P + \eta \; \nabla^2\vec{v}_d, \quad (17a)$$

in the bulk system.

Inserting equation (29) into the last equation, one obtains the equation of motion for $\vec{v}_d$:

$$\frac{\partial \vec{v}_d}{\partial t} + (\vec{v}_d \cdot \nabla)\vec{v}_d = \frac{\rho_c}{\rho_d} \nabla \left[\frac{-\hbar^2}{8m^2} \frac{(\nabla \rho_c)^2}{\rho_c^2} + \frac{\hbar^2}{4m^2} \frac{\nabla^2 \rho_c}{\rho_c} + \tilde{\mu}(\rho_c, \rho_d) \right.$$

$$\left. -\alpha \frac{\rho_d}{2\rho} (\vec{v}_d - \vec{v}_c)^2 \right] - \frac{\nabla P}{\rho_d} + \eta \frac{\nabla^2 v}{\rho_d} d \tag{35}$$

One can also obtain the equation of motion for the momentum density $\vec{j}$, from equation (26), $\vec{j} = \rho_c \vec{v}_c + \rho_d \vec{V}_d$ and equations (29) - (35),

$$\frac{\partial \vec{j}}{\partial t} + \nabla \cdot \overleftrightarrow{P} = \frac{\alpha}{2\rho} \left[\nabla \cdot \left\{ \rho_c \rho_d (\vec{v}_d - \vec{v}_c) \right\} \right] (\vec{v}_d - \vec{v}_c) + \eta \nabla^2 \vec{v}_d, \tag{36}$$

where

$$\nabla \cdot \overleftrightarrow{P} = \left[\rho_c (\vec{v}_c \cdot \nabla)\vec{v}_c + \rho_d (\vec{v}_d \cdot \nabla)\vec{v}_d + \{\nabla \cdot (\rho_c \vec{v}_c)\}\vec{v}_c + \{\nabla \cdot (\rho_d \vec{v}_d)\}\vec{v}_d \right] + \nabla P. \tag{37}$$

To find the equation for the energy conservation law, one can first obtain the energy density ϵ per unit mass, from equations (14), (19) - (28), as follows :

$$\epsilon = \frac{1}{2}\rho_c v_c^2 + \frac{1}{2}\rho_d v_d^2 - \frac{(\rho_c \rho_d)}{2\rho}(\vec{v}_d - \vec{v}_c)^2 + \xi_0(\rho_c, \rho_d), \tag{38}$$

where

$$\beta = \frac{m \rho}{3\hbar^2} \int V(r) \mathcal{S}_2^2(r) h(r) r^2 d^3 r, \tag{39}$$

and

$$\tilde{\epsilon}_0 = \frac{\rho_c^2}{2m} \int V(r) \mathcal{S}_1^2(r) d^3 r + \frac{\rho_c \rho_d}{m} \int V(r) \mathcal{S}_2^2(r) \left[1 + h(r)\right] d^3 r$$

$$- \frac{\hbar^2}{2m} \lim_{\vec{r} \to 0} \nabla_{\vec{r}}^2 \left| \Omega_1(\vec{r}, \vec{R}) \right| \tag{40}$$

The last term of equation (40) is the 'internal' kinetic energy density which is independent of the macroscopic flow velocities. We have user $\vec{r} = \vec{X}_1 - \vec{x''}_1$, $\vec{R} = \frac{1}{2}(\vec{x'} + \vec{x''}_1)$, and only terms up to quadratic in the velocity difference $\vec{v}_d - \vec{v}_c$ terms are retained. After taking the time derivative of the energy density ϵ in equation (38), the trivial equation for the energy conservatron law can be obtained, but the derivation will not be presented here.

General Discussion and Conclusion

Equations (10), (12), (15), and (17) which have been presented in the second section are just the usual hydrodynamic equations involving the interacting potential. In the last section we have obtained equations (29), (30), (31), (32), (35), and (36). They are not yet a complete set of thermo-hydrodynamic equations of HeII; more investigations are needed on the equations of motion for the entropy density and energy density. It is obscure whether the condensate density or the depletion density carries the entropy. Derivation of these equations was unsuccessfully attempted ten years ago[12]. However we have attempted to derive a

closed set of thermo-hydrodynamic equations which the numerical values of $\tilde{\mu}$, α, and β can be estimated from the interacting potential V(r), such as the Lennard-Jones (12-6) potential or the HFDHE2 potential [15]. It has been believed widely over thirty years that the appropriate potential for the description of the interaction between neutral helium atoms is the L-J (12-6) potential proposed by the Boer and Michels[16]. Therefore other microscopic theories of superfluid HeII, such as the Bogolubov model[17] which involves the Fourier transform of the interacting pototential, give divergent results of Fourier Transform of L-J (12-6) potential into momentum space. But the L-J (12-6) potential may not be the appropriate potential for He II system as discussed by Bogolubov and Kurbutov[18]. The HFDHE^2 potential proposed for helium by Aziz *et al.*[15] will serve the Bogolubov model as well as suggested by H.R. Glyde (private communication).

Our discussion will not be completed unless we mention Landau's successful microscopic theory [19]. Landau's theory was based on a two-fluid model with the normal density ρ_n and the superfluid density ρ_s, where the normal density arose from the elementary excitations or the quasi-particles as called the phonons and rotons. Although he used the Bose statistics for the phonons and rotons, he never took the Bose-Einstein condensation into account for HeII.

Bogolubov [17] has shown that the Bose-Einstein condensation occurs in a weakly interacting Bose system and has obtained the energy spectrum which agrees with Landau's microscopic theory. Bogolubov has then taken the condensate density ρ_c, the normal density ρ_n and the superfluid density ρ_s for a model of He II. Up to now, the microscopic theories of superfluidity has made a little progress. Bogolubov's microscopic theory also needs more investigation. Even though Landau's microscopic theory is very successful in predicting the existence of the second sound, known as temperature (entropy) waves, and other macroscopic properties of He II, it is still irrelevant to the problem of what causes the superfluidity. Therefore the connection between Landau's two-fluid model and the condensate depletion model was attempted[6, 12], since it may yield the solution of the superfluidity. Our closed set of thermo-hydrodynamic equations is also our attempt to find the solution, and it will hopefully at least yield the expression for the velocity of second sound as well as Landau's microscopic theory.

Acknowledgement

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    ได้พยายามหาสมการเทอร์โมไฮโดรไดนามิกชุดที่สมบูรณ์ชุดหนึ่ง   สำหรับส่วนควบแน่นและไม่ควบ
แน่น ของฮีเลียมเหลวชนิดที่สอง ในภาชนะขนาดโต โดยใช้ทฤษฎีจุลภาคที่ได้พัฒนามาจากการอธิบายว่าการควบแน่น
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